Partial Fractions

$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$ //

Integration of Rational Functions...
Examples,

- $\int \frac{1}{x}\,dx = \ln|x|+C, \int \frac{1}{x^2+1}\,dx = \tan^{-1}(x) + C$
- Substitution Rule and the above information...
  $\int \frac{1}{2x+1}\,dx \overset{?}{=} \int \frac{1}{3+4x^2}\,dx = \frac{1}{4}\int \frac{1}{\frac{3}{4}+x^2}dx = \frac{1}{4}\int \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2+x^2}dx = \frac{1}{4}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x}{\frac{\sqrt{3}}{2}}\right) = \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2x}{\sqrt{3}}\right)$

  ~~$u = 3+4x^2$~~
  ~~$du = 8x\,dx$~~

  $u = 2x+1$
  $du = 2\,dx$

  $= \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|2x+1| + C$
- By recognizing a rational function as sum of terms....
  $\int \frac{1}{x^2-1}\,dx = \int \frac{1}{(x+1)(x-1)}dx = \int\left[\frac{1}{2}\cdot\frac{1}{(x-1)} - \frac{1}{2(x+1)}\right]dx$

Partial fractions is an algebraic method that reveres the addition of rational functions, and can be used to integrate any rational function.
*general factorization skills are important here!

Degree of numerator < Degree of denominator

Suppose that $N(x)$ and $D(x)$ are polynomials. The basic strategy is to be able to write the rational function $R(x) = \frac{N(x)}{D(x)}$ as a sum of very simple, easy to integrate rational functions such as

- CASE I: Denominator factors into a number of distinct linear factors $(x-1)(x+5)$
- CASE II: Denominator factors into a number of linear factors that are not distinct (i.e., repeated linear factors) $\rightarrow (x+1)^2$ or $(2x+5)^3$
- CASE III: Denominator factors into a number of linear factors and distinct irreducible quadratic factors $\rightarrow (x+1)(2x+3)(x^2+x+8)$

In general the simplest integrals we will get in the above threes cases will have the following forms:

$$\int \frac{A}{x-a}\,dx \qquad \int \frac{A}{(x-a)^n}\,dx \qquad \int \frac{Ax+b}{ax^2+bx+c}\,dx$$

CASE I $\longrightarrow$ Distinct linear factors as denominator

$\boxed{\text{Ex:}}$ $\displaystyle\int \frac{x-3}{x^2-3x+2}\,dx = \int\left[\frac{A}{x-1}+\frac{B}{x-2}\right]dx$

$$= \int\left[\frac{2}{x-1}+\frac{(-1)}{x-2}\right]dx$$

$$= 2\int\frac{1}{x-1}\,dx - 1\int\frac{1}{x-2}\,dx$$

$$= 2\ln|x-1| - \ln|x-2| + c$$

$$= \ln|x-1|^2 - \ln|x-2| + c$$

$$= \ln\frac{|x-1|^2}{|x-2|} + c \quad //$$

$x^2-3x+2 = (x-1)(x-2)$

$\left.\dfrac{x-3}{x^2-3x+2} = \dfrac{A}{x-1} + \dfrac{B}{x-2}\right]$ LCD

$\dfrac{x-3}{\cancel{(x^2-3x+2)}} = \dfrac{A(x-2)+B(x-1)}{\cancel{(x-1)(x-2)}}$

$x-3 = Ax-2A+Bx-B$

$1.x-3 = (A+B)x+(-2A-B)$

$\left.\begin{aligned}A+B&=1\\-2A-B&=-3\end{aligned}\right]$ solve for $A, B \longrightarrow \boxed{\begin{aligned}A&=2\\B&=-1\end{aligned}}$

Example

$\boxed{\text{Ex:}}$ $\displaystyle\int \frac{x^2+1}{x^3-2x^2-3x}\,dx = \int\left[\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-3}\right]dx$

$$= -\frac{1}{3}\int\frac{1}{x}\,dx + \frac{5}{6}\int\frac{1}{x+1}\,dx + \frac{1}{2}\int\frac{1}{x-3}\,dx$$

$$= -\frac{1}{3}\ln|x| + \frac{5}{6}\ln|x+1| + \frac{1}{2}\ln|x-3| + C \quad //$$

$x^3-2x^2-3x = x(x^2-2x-3)$

$= x(x+1)(x-3)$

$\left.\dfrac{x^2+1}{x^3-2x^2-3x} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-3}\right]$ LCD

$x^2+1 = A(x+1)(x-3) + Bx(x-3) + Cx(x+1)$

$x^2+1 = A(x^2-3x+x-3) + B(x^2-3x) + C(x^2+x)$

$x^2+1 = A(x^2-2x-3) + (Bx^2+Cx^2) + (-3Bx+Cx)$

$x^2+1 = (A+B+C)x^2 + (-2A-3B+C)x + (-3A)$

$\left.\begin{aligned}&A+B+C=1;\quad -2A-3B+C=0\\&-3A=1 \Rightarrow A=-1/3\end{aligned}\right]$ Solve for A, B, C $\rightarrow$ $\boxed{\begin{aligned}A&=-1/3\\B&=5/6\\C&=1/2\end{aligned}}$

CASE II repeated linear terms as denominator

Ex: $\displaystyle\int \frac{x-1}{x^2(x+2)}\,dx = \int\left[\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}\right]dx$

$x^2(x+1) \leftarrow$ ~~Quadratic factor~~

repeated linear ($x \cdot x$)

distinct linear

$$\frac{x-1}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$

$$x-1 = Ax(x+2) + B(x+2) + Cx^2$$

$$x-1 = Ax^2 + 2Ax + Bx + 2B + Cx^2$$

$$x-1 = (A+C)x^2 + (2A+B)x + 2B$$

$$\left.\begin{aligned} A+C=0 &\Rightarrow C=-A=-3/4 \\ 2A+B=1 &\Rightarrow A=\frac{1-B}{2}=\frac{3}{4} \\ 2B=-1 &\Rightarrow B=-1/2 \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} A&=3/4 \\ B&=-1/2 \\ C&=-3/4 \end{aligned}}$$

$$= \left[\frac{3}{4}\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{dx}{x^2} - \frac{3}{4}\int \frac{1}{x+2}\,dx\right]$$

$$= \frac{3}{4}\ln|x| - \frac{1}{2}\left(-\frac{1}{x}\right) - \frac{3}{4}\ln|x+2| + C$$

$$= \frac{3}{4}\left(\ln|x| - \ln|x+2|\right) + \frac{1}{2x} + C$$

$$= \frac{3}{4}\ln\left|\frac{x}{x+2}\right| + \frac{1}{2x} + C \;//$$

Example

Ex: $\displaystyle\int \frac{1}{(x^2-1)^2}\,dx = \int\left[\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}\right]dx$

$(x^2-1)^2 = (x+1)^2(x-1)^2 \leftarrow$ repeated linear

$$\frac{1}{(x^2-1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$1 = A(x+1)(x-1)^2 + B(x-1)^2 + C(x+1)^2(x-1) + D(x+1)^2$$

$\downarrow$

Solve for A, B, C, D

$\downarrow$

$A = \frac{1}{4}$, $B = \frac{1}{4}$, $C = -\frac{1}{4}$, $D = \frac{1}{4}$

$$= \frac{1}{4}\int \frac{dx}{x+1} + \frac{1}{4}\int \frac{dx}{(x+1)^2} - \frac{1}{4}\int \frac{dx}{x-1} + \frac{1}{4}\int \frac{dx}{(x-1)^2}$$

$$= \frac{1}{4}\ln|x+1| + \frac{1}{4}\left(-\frac{1}{x+1}\right) - \frac{1}{4}\ln|x-1| + \frac{1}{4}\left(-\frac{1}{x-1}\right) + C$$

$$= \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{1}{4}\left(\frac{1}{x+1} - \frac{1}{x-1}\right) + C$$

$$= \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| + \frac{1}{2(x^2-1)} + C \;//$$

CASE III distinct linear + irreducible quadratic

Ex: $\int \frac{1}{(x-1)(x^2+5)}dx = \int\left[\frac{A}{x-1} + \frac{Bx+C}{x^2+5}\right]dx$

$\frac{1}{(x-1)(x^2+5)} = \frac{A}{(x-1)} + \frac{Bx+C}{x^2+5}$

$1 = A(x^2+5) + (Bx+C)(x-1)$

$1 = Ax^2+5A + Bx^2 - Bx + Cx - C$

$1 = (A+B)x^2 + (-B+C)x + (5A-C)$

0 0

$A+B=0$, $-B+C=0$, $5A-C=1$ → Solve for A, B, C → $\boxed{A = 1/6,\ B = -1/6,\ C = -1/6}$

$= \frac{1}{6}\int \frac{1}{x-1}dx - \frac{1}{6}\int \frac{x+1}{x^2+5}dx$

$= \frac{1}{6}\ln|x-1| - \frac{1}{6}\int \frac{x\,dx}{x^2+5} - \frac{1}{6}\int\frac{1}{x^2+5}dx$

$= -\frac{1}{6}\cdot\frac{1}{2}\int\frac{du}{u} - \frac{1}{6}\int\frac{1}{x^2+(\sqrt{5})^2}dx$

$= -\frac{1}{12}\ln|u| - \frac{1}{6}\cdot\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$

$= \frac{1}{6}\ln|x-1| - \frac{1}{12}\ln|x^2+5| - \frac{1}{6\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$ //

Example

Ex: $\int \frac{x-1}{(x+2)(3x^2+1)}dx = \int\left[\frac{A}{x+2} + \frac{Bx+C}{3x^2+1}\right]dx$

$\frac{x-1}{(x+2)(3x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{3x^2+1}$

$x-1 = A(3x^2+1) + (Bx+C)(x+2)$

$x-1 = 3Ax^2 + A + Bx^2 + 2Bx + Cx + 2C$

$1.x-1 = (3A+B)x^2 + (2B+C)x + (A+2C)$

$3A+B=0$, $2B+C=1$, $A+2C=-1$ → Solve for A, B, C → $\boxed{A = -3/13,\ B = 9/13,\ C = -5/13}$

$= \left(\frac{-3}{13}\right)\int\frac{dx}{x+1} + \frac{9}{13}\int\frac{x\,dx}{3x^2+1} + \left(\frac{-5}{13}\right)\int\frac{dx}{3x^2+1}$

$= -\frac{3}{13}\ln|x+1| + \frac{9}{13}\cdot\frac{1}{6}\int\frac{du}{u}$ ($u=3x^2+1$, $du = 6x\,dx$) $- \frac{5}{13}\cdot\frac{1}{3}\int\frac{dx}{x^2+\left(\frac{1}{\sqrt{3}}\right)^2}$

$= -\frac{3}{13}\ln|x+1| + \frac{3}{26}\ln|u| - \frac{5}{39}\cdot\frac{1}{1/\sqrt{3}}\tan^{-1}(\sqrt{3}x) + C$

$= -\frac{3}{13}\ln|x+1| + \frac{3}{26}\ln|3x^2+1| - \frac{5\sqrt{3}}{39}\tan^{-1}(\sqrt{3}x) + C$ //

Practice Examples

$x = e^t \to dx = e^t dt \to \frac{dx}{x} = dt$

$\boxed{\text{Ex:}}$ $\displaystyle\int \frac{1}{e^{2t}+e^t+1}\,dt = \int \frac{1}{x^2+x+1}\cdot\frac{dx}{x}$

$= \displaystyle\int \frac{1}{x(x^2+x+1)}\,dx.$

$\dfrac{1}{x(x^2+x+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+x+1}$

$1 = A(x^2+x+1) + (Bx+C)\cdot x$

$1 = Ax^2 + Ax + A + Bx^2 + Cx$

$1 = (A+B)x^2 + (A+C)x + A$

$A = 1$

$A+B=0 \Rightarrow B=-A=-1$

$A+C=0 \Rightarrow C=-A=-1$

$= \displaystyle\int\left(\frac{1}{x} + \frac{-x-1}{x^2+x+1}\right)dx$

$= \displaystyle\int \frac{1}{x}\,dx - \int\frac{(x+1)\,dx}{x^2+x+1}$

$= \ln|x| - \displaystyle\int\frac{(x+\frac{1}{2}+\frac{1}{2})\,dx}{x^2+x+1}$ $\quad u = x^2+x+1$, $du = (2x+1)dx$

$= \ln|x| - \displaystyle\int\frac{(x+\frac{1}{2})\,dx}{x^2+x+1} - \frac{1}{2}\int\frac{dx}{x^2+x+1}$ sum of square

$= \ln|x| - \dfrac{1}{2}\displaystyle\int\frac{du}{u} - \frac{2}{3}\int\frac{dx}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}$

Practice Examples

$= \ln|x| - \frac{1}{2}\ln|x^2+x+1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

$= \ln|e^t| - \frac{1}{2}\ln|e^{2t}+e^t+1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2e^t+1}{\sqrt{3}}\right) + C$

$= t -$

$\boxed{\text{Ex:}}$ $\displaystyle\int \frac{5x+1}{x^2+5x+6}\,dx$

$x^2+5x+6 = (x+2)(x+3)$

$\dfrac{5x+1}{(x+2)(x+3)} = \dfrac{A}{x+2} + \dfrac{B}{x+3}$

$5x+1 = A(x+3) + B(x+2)$

$5x+1 = (A+B)x + (3A+2B)$

$\left.\begin{aligned} A+B&=1\\ 3A+2B&=5\end{aligned}\right\} \to \boxed{\begin{aligned} A&=-9\\ B&=14\end{aligned}}$

$\displaystyle\int\frac{-9\,dx}{x+2} + \int\frac{14\,dx}{x+3}$

$= -9\ln|x+2| + 14\ln|x+3| + C$

Degree of numerator $\geq$ Degree of denominator

CASE IV: Method of long division followed by CASE (I/II/III)

Goal $\rightarrow$ $\dfrac{N(x)}{D(x)} = P(x) + \dfrac{R(x)}{D(x)}$ remainder of long div.

Deg. $R(x) <$ Deg $D(x)$

$\boxed{\text{Ex:}}$ $\displaystyle\int \frac{4x^2 - 3x + 2}{(4x^2 - 4x + 3)}\, dx$

$$\begin{array}{r|l} & \quad 1 \\ 4x^2-4x+3 & \; 4x^2-3x+2 \\ & -\;\,4x^2-4x+3 \\ & \quad\;\, - \;\; + \;\; - \\ \hline & \qquad x-1 \end{array}$$

$x-1$ $\leftarrow$ remainder

$$4x^2-3x+2 = 1\cdot(4x^2-4x+3) + (x-1)$$

Integrand $\Rightarrow$ $\dfrac{4x^2-3x+2}{4x^2-4x+3} = 1\cdot\dfrac{\cancel{(4x^2-4x+3)}}{\cancel{(4x^2-4x+3)}} + \dfrac{(x-1)}{(4x^2-4x+3)}$

$\Rightarrow$ Integrand $\longrightarrow \left(1 + \dfrac{x-1}{4x^2-4x+3}\right)$

Example

$$\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx = \int\left(1+\frac{x-1}{4x^2-4x+3}\right)dx$$

$$= \int dx + \frac{1}{4}\int \frac{(u-1)\,du}{u^2+2}$$

$$= x + \frac{1}{4}\int \frac{u\,du}{u^2+2} - \frac{1}{4}\int\frac{du}{u^2+2}$$

$$= x + \frac{1}{4}\cdot\frac{1}{2}\ln|u^2+2| - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$= x + \frac{1}{8}\ln|4x^2-4x+3| - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C$$

~~$4x^2-4x+3$~~

~~$= 4x^2 - 6x + 2x + 3$~~

~~$= 2x(2x-1) - 3(2x$~~

$4x^2-4x+3$

$= (2x-1)^2+2$

$= u^2+2$

$u = 2x-1$

$du = 2dx$

$x-1 = \dfrac{u-1}{2}$

Example

$\boxed{\text{Ex:}}$ Evaluate $\displaystyle\int \frac{3x^3-8x^2+4x-1}{x^2-3x+2}\,dx$

$$\frac{3x^3-8x^2+4x-1}{x^2-3x+2} = (3x+1) + \frac{(x-3)}{x^2-3x+2}$$ ← using long division

$$\int (3x+1)\,dx + \int \frac{(x-3)}{(x^2-3x+2)}\,dx$$

$$= 3\cdot\frac{x^2}{2} + x + \int \frac{x-3}{(x-1)(x-2)}\,dx.$$

$$\frac{x-3}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$x-3 = A(x-2) + B(x-1)$$

$$1\cdot x - 3 = (A+B)x + (-2A-B)$$

$$A+B = 1$$
$$-2A-B = -3$$
→ Solve

$$A = 2$$
$$B = -1$$

$$= \frac{3}{2}x^2 + x + \int \left[\frac{2}{x-1} - \frac{1}{x-2}\right] dx$$

Example

$$= \frac{3}{2}x^2 + x + 2\int \frac{dx}{x-1} - \int \frac{dx}{x-2}$$

$$= \frac{3}{2}x^2 + x + 2\ln|x-1| - \ln|x-2| + C$$

Long Division:

$$\begin{array}{r} 3x+1 \\ x^2-3x+2 \overline{\big) \, 3x^3-8x^2+4x-1} \\ \underline{-(3x^3-9x^2+6x)} \\ x^2-2x-1 \\ \underline{-(x^2-3x+2)} \\ (x-2) \end{array}$$

← remainder

$$\boxed{3x^3-8x^2+4x-1 = (3x+1)(x^2-3x+2) + (x-2)}$$

Practice Example

$\boxed{\text{Ex:}}$ $\displaystyle\int \left(3x + \frac{3x+1}{x^2+5} + \frac{3x}{(x^2+5)^2}\right) dx$

$$= 3\int x\,dx + \int \frac{3x}{x^2+5}\,dx + \int \frac{1}{x^2+5}\,dx + \int \frac{3x}{(x^2+5)^2}\,dx$$

$$= 3\cdot\frac{x^2}{2} + \int \frac{dx}{x^2+5} + \int\left[\frac{3x}{x^2+5} + \frac{3x}{(x^2+5)^2}\right]dx$$

$u = x^2+5$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$\frac{3}{2}du = 3x\,dx$

$$= \frac{3}{2}x^2 + \int \frac{dx}{x^2+(\sqrt{5})^2} + \int \frac{3/2}{u} + \frac{3/2}{u^2}\,du$$

$$= \frac{3}{2}x^2 + \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + \frac{3}{2}\int\frac{du}{u} + \frac{3}{2}\int\frac{du}{u^2}$$

$$= \frac{3}{2}x^2 + \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + \frac{3}{2}\ln|u| + \frac{3}{2}\left(-\frac{1}{u}\right) + C$$

Example

$$= \quad\downarrow \quad + \quad \downarrow \quad + \frac{3}{2}\ln|x^2+5| - \frac{3}{2(x^2+5)} + C$$

Chapter 7: Techniques of Integration

- Integration by parts
- Trigonometric Integrals
- Trigonometric Substitution
- Integration with Partial Fractions
- Improper Integrals

Improper Integrals

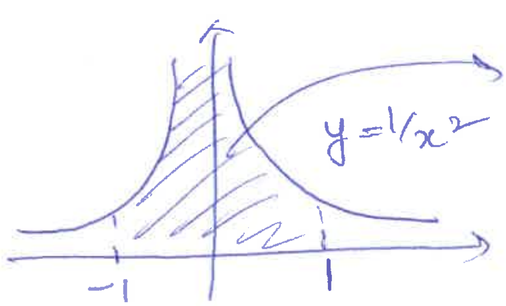
An integral having either an infinite limit of integration or an unbounded integrand is called an improper integral.

FTC has a number of conditions that need to be satisfied before it can be applied. In particular, the function needs to be continuous on the interval. What if it is not? Functions can be unbounded, discontinuous, piecewise, undefined. For example,

$$\int_{-1}^{1} \frac{1}{x^2}\, dx = \left| -\frac{1}{x} \right|_{-1}^{1} = (-1-1) = -2$$

conflict! why?

expected area calculation $\to +\infty$



Infinite Domain of Integration

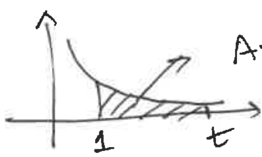
- TYPE I: INFINITE INTERVALS

$\int_a^\infty$, $\int_{-\infty}^b$, $\int_{-\infty}^\infty$

If $\int_a^t f(x)\,dx$ exists for all $t > a$, then we define

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

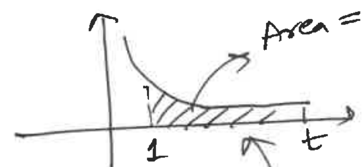
- If the limit exists, we say that $\int_a^\infty f(x)\,dx$ **converges**.
- If the limit does not exist, we say that $\int_a^\infty f(x)\,dx$ **diverges**.



Area $= 1 - \frac{1}{t}$

Area < 1

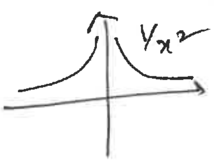
Area $= \ln t$



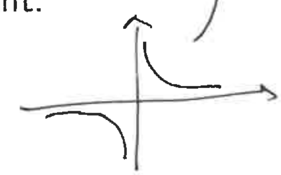
Examples

Ex: Determine whether the integral is convergent or divergent.

infinite domain.



(i) $\int_1^\infty \frac{1}{x^2}\,dx$, (ii) $\int_1^\infty \frac{1}{x}\,dx$



$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$$

$$= \lim_{t\to\infty} \left[-\frac{1}{x} \Big|_1^t \right]$$

$$= \lim_{t\to\infty} \left(-\frac{1}{t} + 1 \right)$$

$$= 0 + 1$$

$\Rightarrow 1$ // Converges.

$$\int_1^\infty \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} \left[\ln|x| \right]_1^t$$

$$= \lim_{t\to\infty} \left(\ln t - \cancel{\ln 1}^{\,0} \right)$$

$$= \lim_{t\to\infty} (\ln t) \longrightarrow \infty$$

Diverges.

the value of $\frac{1}{x}$ does not decrease fast enough to have a finite value.

Improper Integral with Infinite Domain of Integration

* Region for the area is indefinite in the vertical

- If $\int_a^t f(x)\,dx$ exists for all $t > a$, then we define

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

- If $\int_t^b f(x)\,dx$ exists for all $t < b$, then we define

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

- If both $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{a} f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

Examples

$\boxed{\text{Ex:}}$ Evaluate

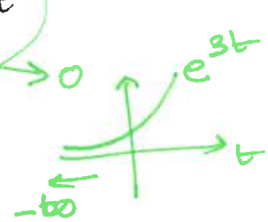
(i) $\int_{-\infty}^{1} e^{3x}\,dx$

$$\int_{-\infty}^{1} e^{3x}\,dx = \lim_{t\to-\infty}\int_t^1 e^{3x}\,dx$$

$$= \lim_{t\to-\infty}\left|\frac{e^{3x}}{3}\right|_t^1$$

$$= \frac{1}{3}\lim_{t\to-\infty}\left(e^3 - e^{3t}\right)$$

$$= \frac{1}{3}e^3 - \frac{1}{3}\lim_{t\to-\infty} e^{3t} \to 0$$



$$= \frac{1}{3}e^3 - 0 = \frac{1}{3}e^3$$

P

(ii) $\int_0^{\infty} xe^{-x}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x}\,dx.$

$$\int_0^t xe^{-x}\,dx = \left[-xe^{-x}\right]_0^t - \int_0^t 1.(-e^{-x})\,dx$$

$$\left.\begin{array}{ll} u=x & dv=e^{-x}dx \\ du=1.dx & v=-e^{-x} \end{array}\right\}$$

$$= \left(-te^{-t}+0\right) + \int_0^t e^{-x}\,dx$$

$$= -te^{-t} + \left|-e^{-x}\right|_0^t$$

$$= -te^{-t} - e^{-t} + e^0$$

L'hospital's Rule

$$\int_0^x xe^{-x}\,dx = \lim_{t\to\infty}\left(\underbrace{-te^{-t}}_{\infty\cdot 0\,\to\,0}\underbrace{-e^{-t}}_{\to 0}+1\right)$$

$$= 1 \,//$$

Example - $\int \frac{1}{x^p}\,dx$

$\boxed{\text{Ex:}}$ For what value of $p(>0)$ is the integral $\int_0^\infty \frac{1}{x^p}\,dx$ convergent?

$$\int_0^\infty \frac{1}{x^p}\,dx = \int_0^1 \frac{1}{x^p}\,dx + \int_1^\infty \frac{1}{x^p}\,dx$$

Infinite Domain

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \frac{1}{1-p}\left[\left(\lim_{t\to\infty}\frac{1}{t^{p-1}}\right)-1\right]$$

Case I: $p>1 \Rightarrow p-1>0 \Rightarrow$ as $t\to\infty$, $t^{p-1}\to\infty$, $\frac{1}{t^{p-1}}\to 0$

So $\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{1-p}(-1) = \frac{1}{p-1}$ //

Case II: $p<1 \Rightarrow p-1<0 \Rightarrow$ as $t\to\infty$, $\frac{1}{t^{p-1}} = t^{1-p}\to\infty$

So $\int_1^\infty \frac{1}{x^p}\,dx \to \infty$ //

Example - $\int \frac{1}{x^p}\,dx$

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{converges to } \frac{1}{p-1}, & p>1 \\ \text{diverges to } \infty, & p\le 1 \end{cases} \leftarrow \left(\int_1^\infty \frac{1}{x}\,dx \to \text{diverges}\right)$$

Unbounded Integrals

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x^p}\,dx$$

Case I: $p>1 \rightarrow \lim_{t\to0^+}\int_t^1 \frac{1}{x^p}\,dx = \lim_{t\to0^+}\frac{1}{1-p}\left(x^{-p+1}\right)_t^1 = \lim_{t\to0^+}\frac{1}{1-p}\left(1-t^{1-p}\right) \to +\infty$

$1-p<0$

As $t\to 0^+$, $t^{1-p}\to\infty$

Case II: $p=1$, $\lim_{t\to0^+}\int_t^1 \frac{1}{x}\,dx = \lim_{t\to0^+}\left(\ln|x|\right)_t^1 = \lim_{t\to0^+}(\ln 1 - \ln t) \to +\infty$

Case III: $p<1$, $1-p>0$, as $t\to0^+$, $t^{1-p}\to 0$, $\lim_{t\to0^+}\int_t^1 \frac{1}{x^p} \to \frac{1}{1-p}$ //

$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{1-p}, & p<1 \\ \text{diverges to } \infty, & p\ge 1 \end{cases}$$

Practice Examples

$\boxed{\text{Ex:}}$ Evaluate

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

$$= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \left(\lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2}\,dx\right) + \left(\lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx\right)$$

$$= \lim_{t\to-\infty} \left| \tan^{-1} x \right|_{t}^{0} + \lim_{t\to\infty} \left| \tan^{-1} x \right|_{0}^{t}$$

$$= \tan^{-1}(0) - \tan^{-1}(-\infty) + \tan^{-1}(\infty) - \tan^{-1}(0)$$

$$= 0 - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - 0$$

$$= \pi$$

(ii) $\displaystyle\int_{4}^{\infty} \frac{1}{x^2-5x+6}\,dx$

$x^2-5x+6 = (x-2)(x-3)$ ← Factorization

$\dfrac{1}{x^2-5x+6} = \dfrac{1}{x-3} - \dfrac{1}{x-2}$ ← Partial fraction

$$\int_4^{\infty} \frac{1}{x^2-5x+6}\,dx$$

$$= \lim_{t\to\infty} \int_4^t \left(\frac{1}{x-3} - \frac{1}{x-2}\right) dx$$

$$= \lim_{t\to\infty} \left| \ln|x-3| - \ln|x-2| \right|_4^t$$

$$= \lim_{t\to\infty} \left(\ln\left|\frac{x-3}{x-2}\right| \right)_4^t$$

$$= \lim_{t\to\infty} \ln\left|\frac{t-3}{t-2}\right| - \ln\left|\frac{1}{2}\right|$$

$$= \lim_{t\to\infty} \ln\left|\frac{1-3/t}{1-2/t}\right| - \ln\left(\frac{1}{2}\right)$$

with $3/t \to 0$ and $2/t \to 0$,

$$= \ln 1 - \ln(1/2)$$

with $\ln 1 \to 0$,

$$= -\ln(1/2).$$

Unbounded Integrands

- TYPE II: DISCONTINUOUS INTEGRALS

  - If f is continuous on $[a, b)$ but discontinuous at $x = b$, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

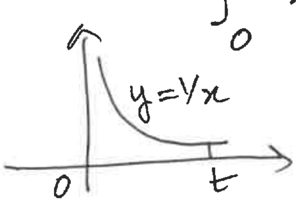
* Situations with Vertical Asymptotes

* Area bounded is infinite in the vertical direction

  - If f is continuous on $(a, b]$ but discontinuous at $x = a$, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

Example, $\displaystyle\int_0^1 \frac{1}{x}\,dx$, $\dfrac{1}{x}$ is continuous on $(0, t]$



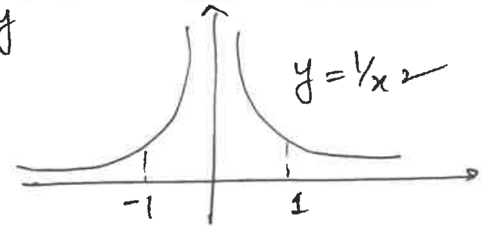
$$\lim_{t\to0^+} \int_t^1 \frac{1}{x}\,dx = \lim_{t\to0^+} \left| \ln x \right|_t^1$$

$$= \lim_{t\to0^+} (\ln 1 - \ln t) \to +\infty$$

with $\ln 1 \to 0$ and $\ln t \to -\infty$.

Example

$\boxed{\text{Ex:}}$ Evaluate $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$ ← Discussed Previously ← FTC fails!



$$\int_{-1}^{1} \frac{1}{x^2}\,dx$$

$$= \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$$

$$= \lim_{a\to 0^-} \int_{-1}^{a} \frac{1}{x^2}\,dx + \lim_{b\to 0^+} \int_{b}^{1} \frac{1}{x^2}\,dx$$

$$= \lim_{a\to 0^-} \left| -\frac{1}{x} \right|_{-1}^{a} + \lim_{b\to 0^+} \left| -\frac{1}{x} \right|_{b}^{1}$$

$$= \lim_{a\to 0^-} \left(-1 - \underbrace{\frac{1}{a}}_{\to +\infty}\right) + \lim_{b\to 0^+} \left(\underbrace{\frac{1}{b}}_{\to +\infty} - 1\right) \longrightarrow +\infty$$

Diverges.

Examples

$\boxed{\text{Ex:}}$ Evaluate

(i) $\displaystyle\int_{0}^{1} \frac{1}{x}\,dx$

$$= \lim_{a\to 0^+} \int_{a}^{1} \frac{1}{x}\,dx$$

$$= \lim_{a\to 0^+} \left| \ln|x| \right|_{a}^{1}$$

$$= \lim_{a\to 0^+} \left(\ln 1 - \ln|a|\right)$$

$$= \lim_{a\to 0^+} -\ln|a|$$

$$= -(-\infty) \longrightarrow +\infty$$

(ii) $\displaystyle\int_{0}^{1} \ln x\,dx$ ← $x \neq 0$

Integration by Parts, $\boxed{\int \ln x\,dx = x\ln x - x}$

$$\int_{0}^{1} \ln x\,dx = \lim_{a\to 0^+} \int_{a}^{1} \ln x\,dx$$

$$= \lim_{a\to 0^+} \left| x\ln x - x \right|_{a}^{1}$$

$$= \lim_{a\to 0^+} \left(1\cdot\underbrace{\ln 1}_{0} - 1 - \underbrace{a\ln a}_{0} + \underbrace{a}_{0}\right)$$

$$= -1$$

Practice Examples

Another example, $\int_1^\infty \frac{\sqrt{x}}{x^2+x}\,dx \to$ converges ?

$\boxed{\text{Ex:}}$ Evaluate

(i) $\int_{-\infty}^{\infty} \frac{1}{(x-2)x^2}\,dx$

$(x-2)x^2 = 0 \to x=0 \quad x=2$

$\int_{-\infty}^{\infty} = \int_{-\infty}^{0} + \int_{0}^{2} + \int_{2}^{\infty}$

$\int_{-\infty}^{0} = \int_{-\infty}^{a} + \int_{a}^{0}$ where $a \to 0^+/0^-$

$\int_{0}^{2} = \int_{0}^{b} + \int_{b}^{2}$ where $b \to ?$

(a bit complicated!) $\to$ Do the thought experiment?

How would you approach the solution!

(ii) $\int_{-1}^{1} \frac{1}{x}\,dx = \int_{-1}^{0} \frac{1}{x}\,dx + \int_{0}^{1} \frac{1}{x}\,dx$

$= \lim_{a\to 0^-} \int_{-1}^{a} \frac{1}{x}\,dx + \lim_{b\to 0^+} \int_{b}^{1} \frac{1}{x}\,dx$

$= \lim_{a\to 0^-} \left|\ln|x|\right|_{-1}^{a} + \lim_{b\to 0^+} \left|\ln|x|\right|_{b}^{1}$

$= \lim_{a\to 0^-} \left(\ln|a| - \cancel{\ln|-1|}\right) + \lim_{b\to 0^+}\left(\cancel{\ln 1} - \ln b\right)$

$= \cancel{-\infty + \infty}$

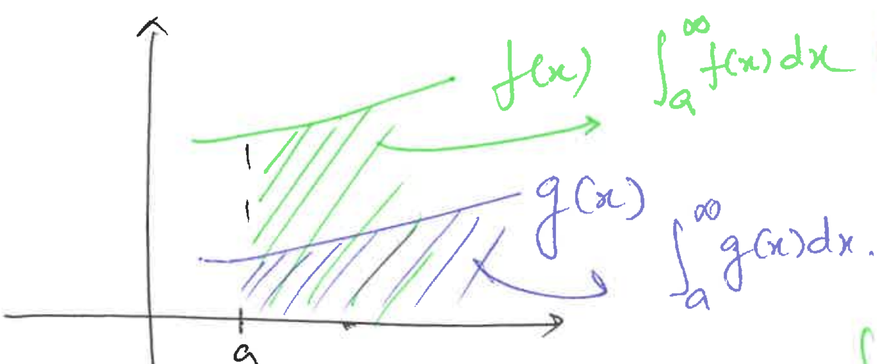
$\to \infty$

Diverges.

Comparison Test for Improper Integrals

Suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

- If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.
- If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.



$f(x) \geq g(x)$ for $x \geq a$

Larger Area $\to$ $\int_a^\infty f(x)\,dx$ converges $\Rightarrow$ $\int_a^\infty g(x)\,dx$ converges.

Smaller Area $\to$ $\int_a^\infty g(x)\,dx$ diverges $\Rightarrow$ $\int_a^\infty f(x)\,dx$ diverges.

Example

$\boxed{\text{Ex:}}$ Show that $\int_1^\infty e^{-x^2}\,dx$ is convergent.

$\int_1^\infty \Rightarrow \quad x \geq 1$

$x^2 \geq x$

$-x^2 \leq -x$

$e^{-x^2} \leq e^{-x}$ for $x \geq 1$

$\uparrow$ $g(x)$ $\quad$ $\uparrow$ $f(x)$

$x \geq 1$: ~~x~~ $\to 2, 4, \frac{7}{3}$

$x^2 \to 4, 16, \frac{49}{9}$

$-x^2 \to -4, -16, -\frac{49}{9}$

$\to$ $-4 < -2$, $-16 < -4$, $-\frac{49}{9} < -\frac{7}{3}$ $\to$ $-x^2 \leq -x$

Prob: $\int_1^\infty f(x)\,dx$ converges?

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_1^t e^{-x}\,dx$$

$$= \lim_{t\to\infty} \Big|-e^{-x}\Big|_1^t$$

$$= \lim_{t\to\infty}\big(e^{-1} - \underbrace{e^{-t}}_{\to 0}\big) = e^{-1} = 1/e \to \text{converges!}$$

Using Comparison Theorem,

since $e^{-x^2} \leq e^{-x}$

and $\int_1^\infty e^{-x}\,dx$ converges

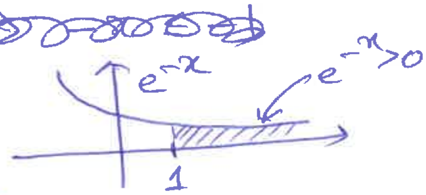
$\Rightarrow \int_1^\infty e^{-x^2}\,dx$ also converges!

Example

$\boxed{\text{Ex:}}$ Does the $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ diverge or converge?

$\int_1^\infty \Rightarrow \quad x \geq 1$

$e^{-x} > 0$



$1 + e^{-x} > 1$

$$\underbrace{\boxed{\frac{1+e^{-x}}{x}}}_{f(x)} > \underbrace{\boxed{\frac{1}{x}}}_{g(x)}$$

We already know,

$\int_1^\infty \frac{1}{x}$ diverges.

(previous slides)

Using Comparison theorem,

$\Rightarrow$ since $\frac{1+e^{-x}}{x} > \frac{1}{x}$ and $\int_1^\infty \frac{1}{x}$ diverges

So $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ also diverges.