

## Useful Trigonometric Information

- Pythagorean Connections:
  - ▶  $\sin^2(x) + \cos^2(x) = 1$
  - ▶  $1 + \tan^2(x) = \sec^2(x)$
  - ▶  $1 + \cot^2(x) = \csc^2(x)$
- Double-angle Formulas:
  - ▶  $\sin(2x) = 2 \sin(x) \cdot \cos(x)$
  - ▶  $\cos(2x) = 1 - 2 \sin^2(x) = 2 \cos^2(x) - 1$
  - ▶  $\Rightarrow \sin^2 x = \frac{1 - \cos 2x}{2}$  ;  $\cos^2 x = \frac{1 + \cos 2x}{2}$
- Triple-angle Formulas:
  - ▶  $\sin(3x) = 3 \sin(x) - 4 \sin^3(x)$
  - ▶  $\cos(3x) = 4 \cos^3(x) - 3 \cos(x)$
- Sum/Diff. Formulas:
  - ▶  $\sin A \cos B = [\sin(A - B) + \sin(A + B)]/2$
  - ▶  $\sin A \sin B = [\cos(A - B) - \cos(A + B)]/2$
  - ▶  $\cos A \cos B = [\cos(A - B) + \cos(A + B)]/2$

$$\star \sin(-\theta) = -\sin \theta$$
$$\cos(-\theta) = \cos \theta$$

## Derivatives of Trigonometric Functions

The derivative rules for trigonometric functions are as follows:

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$
- $\frac{d}{dx}(\csc x) = -\csc x \cdot \cot x$
- $\frac{d}{dx}(\cot x) = -\csc^2 x$

Basic Trig Function Definitions:

- $\tan x = \frac{\sin x}{\cos x}$
- $\sec x = \frac{1}{\cos x}$
- $\csc x = \frac{1}{\sin x}$
- $\cot x = \frac{\cos x}{\sin x}$

## Trigonometric Integrals

There are many such tricks for integrating powers of trigonometric functions. Here we concentrate on two families:

$$\int \sin^m x \cos^n x dx \quad \text{and} \quad \int \tan^m x \sec^n x dx$$

for integer  $n, m$ . The details of the technique depend on the parity of  $n$  and  $m$ , that is, whether  $n$  and  $m$  are even or odd numbers.

$$\boxed{\text{Ex:}} \int \sin^2 x \cos x dx = \int u^2 du$$

u-substitution:

$$u = \sin x$$

$$du = \cos x dx$$

$$= \frac{u^3}{3} + C$$

$$= \frac{1}{3} (\sin x)^3 + C$$

## Integrating $\int \sin^m x \cos^n x dx$

One of  $n$  or  $m$  is odd...

$$\boxed{\text{Ex:}} \int \sin^2 x \cos^3 x dx = \int u^2 \cdot \cos^2 x \cdot du$$

$$u = \sin x$$

$$du = \cos x dx$$

$$\cos^2 x = 1 - \sin^2 x$$

$$= 1 - u^2$$

$$= \int u^2 (1 - u^2) du$$

$$= \int (u^2 - u^4) du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \frac{1}{3} (\sin x)^3 - \frac{1}{5} (\sin x)^5 + C //$$

**NOTE:** This method can be used whenever  $n$  is an odd integer.

- Substitute  $u = \sin x$  and  $du = \cos x dx$
- This leaves an even power of cosines - convert then using  $\cos^2 x = 1 - \sin^2 x = 1 - u^2$

## Example

If  $m$  is odd, then use the same strategy but with the roles reversed.

- Substitute  $u = \cos x$  and  $du = -\sin x dx$
- This leaves an even power of cosines - convert then using  $\sin^2 x = 1 - \cos^2 x = 1 - u^2$

Ex: Evaluate  $\int \sin^3 x \cos^2 x dx$

$$u = \cos x$$

$$du = -\sin x dx$$

$$\begin{aligned} \sin^2 x &= 1 - \cos^2 x \\ &= 1 - u^2 \end{aligned}$$

$$\begin{aligned} &= \int \sin^2 x \cdot \sin x \cdot \cos^2 x dx \\ &= \int (1-u^2) \cdot u^2 \cdot -du \\ &= \int (-u^2 + u^4) du \\ &= -\frac{u^3}{3} + \frac{u^5}{5} + C \\ &= -\frac{1}{3}(\cos x)^3 + \frac{1}{5}(\cos x)^5 + C // \end{aligned}$$

## Trigonometric Integrals

Using Trigonometric identities to express higher powers of sine and cosine functions in terms of lower powers..

Ex: Express in terms of 'smaller' powers:

$$\begin{aligned} \text{(i) } \sin^4(x) &= (\sin^2 x)^2 \\ &= \left( \frac{1 - \cos 2x}{2} \right)^2 \\ &= \frac{1 + \cos^2 2x - 2\cos 2x}{4} \end{aligned}$$

$$= \frac{1}{4} + \frac{1}{4} \cos^2 2x - \frac{1 \cos 2x}{2}$$

$$= \frac{1}{4} + \frac{1}{4} \left( \frac{1 + \cos 4x}{2} \right) - \frac{1 \cos 2x}{2} = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} \cos(4x) - \frac{1 \cos 2x}{2}$$

$$\begin{aligned} \text{(ii) } \cos^4(x) &= (\cos^2 x)^2 \\ &= \left( \frac{1 + \cos 2x}{2} \right)^2 \\ &= \frac{1}{4} (1 + \cos^2 2x + 2\cos 2x) \\ &= \frac{1}{4} \left( 1 + \frac{1 + \cos 4x}{2} + 2\cos 2x \right) \\ &= \frac{3}{8} + \frac{1}{8} \cos 4x + \frac{1}{2} \cos 2x // \end{aligned}$$

$$= \frac{3}{8} + \frac{1}{8} \cos(4x) - \frac{1 \cos 2x}{2}$$

# Integrating $\int \sin^m x \cos^n x dx$

Both  $n$  and  $m$  are even ... Strategy is to use trig. identities (example - half-angle identities) to revert back to first case with odd powers.

**Ex:** Evaluate

$$\begin{aligned}
 \text{(i) } & \int \cos^2 x dx \\
 &= \int \left( \frac{1 + \cos 2x}{2} \right) dx \\
 &= \int \left( \frac{1}{2} + \frac{1}{2} \cos 2x \right) dx \\
 &= \frac{1}{2} \int dx + \frac{1}{2} \int \cos 2x dx \\
 &= \frac{x}{2} + \frac{1}{2} \cdot \frac{\sin 2x}{2} + C \\
 &= \frac{x}{2} + \frac{1}{4} \sin 2x + C //
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii) } & \int \cos^4 x dx \quad \leftarrow \text{from last page} \\
 &= \int \left( \frac{3}{8} + \frac{1}{8} \cos(4x) + \frac{1}{2} \cos(2x) \right) dx \\
 &= \frac{3}{8} \int dx + \frac{1}{8} \int \cos(4x) dx + \frac{1}{2} \int \cos 2x dx \\
 &= \frac{3}{8} x + \frac{1}{8} \cdot \frac{\sin 4x}{2} + \frac{1}{2} \cdot \frac{\sin 2x}{2} + C \\
 &= \frac{3}{8} x + \frac{1}{16} \sin(4x) + \frac{1}{4} \sin(2x) + C //
 \end{aligned}$$

## Example

**Ex:** Evaluate  $\int \sin^2 x \cos^2 x dx$

$$\begin{aligned}
 &= \int \left( \frac{1 - \cos 2x}{2} \right) \left( \frac{1 + \cos 2x}{2} \right) dx \\
 &= \frac{1}{4} \int (1 - \cos 2x)(1 + \cos 2x) dx \quad \leftarrow (A+B)(A-B) = A^2 - B^2 \\
 &= \frac{1}{4} \int (1 - \cos^2 2x) dx \\
 &= \frac{1}{4} \int \left( 1 - \frac{1 + \cos 4x}{2} \right) dx \\
 &= \frac{1}{4} \int \left( 1 - \frac{1}{2} - \frac{\cos 4x}{2} \right) dx \\
 &= \frac{1}{8} \int dx - \frac{1}{8} \int \cos(4x) dx = \frac{1}{8} x - \frac{1}{8} \frac{\sin 4x}{4} + C \\
 &= \frac{x}{8} - \frac{\sin 4x}{32} + C //
 \end{aligned}$$

# Integrating $\int \sin mx \cos nx dx$ form

Using Sum/Diff. Rules of trig. functions...  $\int \sin A \cdot \cos B = \int \frac{\sin(A-B) + \sin(A+B)}{2}$

**Ex:** Evaluate

(i)  $\int \sin 3x \cos 6x dx$

$$= \frac{1}{2} \int \sin(3x-6x) + \sin(3x+6x) dx$$

$$= \frac{1}{2} \int (\sin(-3x) + \sin(9x)) dx$$

~~$$= \frac{1}{2} \int \sin 3x dx$$~~

$$= \frac{1}{2} \int \sin(-3x) dx + \frac{1}{2} \int \sin(9x) dx$$

$$= \frac{1}{2} \cdot \frac{\cos(-3x)}{-3} + \frac{1}{2} \cdot \frac{-\cos(9x)}{9} + C$$

$$= \frac{1}{6} \cos(-3x) - \frac{1}{18} \cos(9x) + C$$

Special Cases

$$= \frac{1}{6} \cos(3x) - \frac{1}{18} \cos(9x) + C //$$

(ii)  $\int \sin 2x \sin 6x dx$

$$= \frac{1}{2} \int [\cos(2x-6x) - \cos(2x+6x)] dx$$

$$= \frac{1}{2} \int [\cos(-4x) - \cos(8x)] dx$$

$$= \frac{1}{2} \int \cos(-4x) dx - \frac{1}{2} \int \cos(8x) dx$$

$$= \frac{1}{2} \int \cos(4x) dx - \frac{1}{2} \int \cos(8x) dx$$

$$= \frac{1}{2} \cdot \frac{\sin(4x)}{4} - \frac{1}{2} \cdot \frac{\sin(8x)}{8} + C$$

$$= \frac{1}{8} \sin(4x) - \frac{1}{16} \sin(8x) + C //$$

**Ex:** Evaluate

(i)  $\int \sec x dx$

$$= \int \frac{\sec x (\sec x + \tan x)}{(\sec x + \tan x)} dx$$

$$= \int \frac{\sec^2 x + \sec x \cdot \tan x}{\sec x + \tan x} dx$$

$$= \int \frac{du}{u}$$

$$= \ln |u| + C$$

$$= \ln |\sec x + \tan x| + C //$$

(ii)  $\int \sec^3 x dx$

$$I = \int \sec^2 x \cdot \sec x dx$$

$$u = \sec x \quad dv = \sec^2 x dx$$

$$\downarrow \quad \downarrow$$

$$du = \sec x \cdot \tan x dx \quad v = \tan x$$

$$I = \sec x \cdot \tan x - \int \tan x \cdot \sec x \cdot \tan x dx$$

$$I = \sec x \cdot \tan x - \int \sec x \cdot \tan^2 x dx$$

$$I = \sec x \cdot \tan x - \int \sec x (1 + \sec^2 x) dx$$

$$I = \downarrow - \int \sec x dx - \int \sec^3 x dx$$

$$I = \downarrow - \int \sec x dx - I$$

$$2I = \sec x \cdot \tan x - \ln |\sec x + \tan x|$$

$$I = (\sec x \cdot \tan x - \ln |\sec x + \tan x|) / 2$$

## Integrating $\int \tan^m x \sec^n x dx$

If  $n$  is even (even power of secant), then use the following strategy:

- Substitute  $u = \tan x$  and  $du = \sec^2 x dx$
- Use the trig. identity,  $1 + \tan^2 x = \sec^2 x$ , to convert the remaining factors of  $\sec x$  in terms of  $\tan x$ .

**Ex:** Evaluate  $\int \sec^4 x dx$

$$= \int (\sec^2 x)^2 dx$$

$$= \int \sec^2 x \cdot \sec^2 x dx$$

$$= \int (1 + \tan^2 x) \cdot \sec^2 x dx$$

$$= \int (1 + u^2) du$$

$$= \int u + \frac{u^3}{3} + C = \tan x + \frac{1}{3}(\tan x)^3 + C //$$

$$u = \tan x$$
$$du = \sec^2 x dx$$

## Example

**Ex:** Evaluate  $\int \tan^3 x \sec^4 x dx$

$$= \int \tan^3 x \cdot (\sec^2 x)^2 dx$$

$$= \int u^3 \cdot (1 + u^2) \cdot du$$

$$= \int (u^3 + u^5) du$$

$$= \frac{u^4}{4} + \frac{u^6}{6} + C$$

$$= \frac{1}{4}(\tan x)^4 + \frac{1}{6}(\tan x)^6 + C //$$

$$u = \tan x \rightarrow 1 + \tan^2 x = \sec^2 x$$
$$\boxed{\sec^2 x = 1 + u^2}$$
$$du = \sec^2 x dx$$

## Integrating $\int \tan^m x \sec^n x dx$

If  $m$  is odd (odd power of tangent), then use the following strategy:

- Substitute  $u = \sec x$  and  $du = \sec x \tan x dx$
- Use the trig. identity convert the remaining factors in terms of  $\sec x$ .  
 $\tan^2 x = \sec^2 x - 1 = u^2 - 1$

**Ex:** Evaluate  $\int \tan^3 x \sec^2 x dx$

$$= \int \tan^2 x \cdot \sec x \cdot (\tan x \cdot \sec x dx)$$

$$\begin{aligned} u &= \sec x \\ du &= \sec x \tan x dx \end{aligned}$$

$$1 + \tan^2 x = u^2$$

$$\tan^2 x = u^2 - 1$$

$$= \int (u^2 - 1) \cdot u \cdot du$$

$$= \int (u^3 - u) du$$

$$= \frac{u^4}{4} - \frac{u^2}{2} + C = \frac{1}{4}(\sec x)^4 - \frac{1}{2}(\sec x)^2 + C //$$

## Special Cases

Sometimes we may need to use trig. identities, integration by parts, and some combination of various trig. rules, to find the integral.

**Ex:** Evaluate

(i)  $\int \sec x dx$

$$= \ln |\sec x + \tan x| + C //$$

(ii)  $\int \tan x dx$

$$= \int \frac{\sin x}{\cos x} dx$$

$$= -\ln |\cos x| + C$$

$$= \ln |\sec x| + C //$$

(iii)  $\int \csc x dx$

$$= \int \frac{\csc x (\csc x + \cot x)}{\csc x + \cot x} dx$$

$$= \int \frac{du}{u} = -\ln |u| + C = -\ln |\csc x + \cot x| + C //$$

(iv)  $\int \cot x dx$

$$= \int \frac{\cos x}{\sin x} dx$$

$$= \int \frac{du}{u} = \ln |\sin x| + C //$$

## Special Cases

$$\begin{aligned}
 \text{(v)} \quad & \int \tan^3 x \, dx \\
 &= \int \tan x (\tan^2 x) \, dx \\
 &= \int \tan x (\sec^2 x - 1) \, dx \\
 &= \int \tan x \cdot \sec^2 x \, dx - \int \tan x \, dx
 \end{aligned}$$

$$\text{(vi)} \quad \int \sec^3 x \, dx$$

↙ integrating by parts  
 ↗ recurring integral approach

$$\begin{aligned}
 \text{(vii)} \quad & \int \tan^4 x \, dx \\
 &= \int (\tan^2 x)^2 \, dx \\
 &= \int (\sec^2 x - 1)^2 \, dx \\
 &= \int (\sec^4 x - 2\sec^2 x + 1) \, dx \\
 &= \int \sec^4 x \, dx - 2 \int \sec^2 x \, dx + \int dx
 \end{aligned}$$

$$\begin{aligned}
 \text{(viii)} \quad & \int \csc^3 x \, dx \\
 &= \tan x + \frac{1}{3} (\tan x)^3 - 2 \tan x + x + C \\
 &= \tan x + \frac{1}{3} \tan^3 x + x + C //
 \end{aligned}$$

↗ last few slides  
 ↖ Alternate Textbook example 1.8.23

or,

$$\int \csc^3 x \, dx = \int \frac{1}{\sin^3 x} \, dx$$

$$= -\frac{1}{2} \cot x \cdot \csc x + \frac{1}{2} \ln |\csc x - \cot x| + C //$$

## Practice Examples

$$\begin{aligned}
 \text{(i)} \quad & \int \sin^5 x \, dx \\
 &= \int \sin^4 x \cdot \sin x \, dx \\
 &= \int (1-u^2)^2 (-du) \\
 &= -\int (1+u^4-2u^2) \, du \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad & \int \frac{\sin^3 x}{\cos^4 x} \, dx \\
 &= \int \frac{\sin^2 x \cdot \sin x \, dx}{\cos^4 x} \\
 &= \int \frac{(1-u^2)(-du)}{u^4} \\
 &= \int \left( \frac{u^2-1}{u^4} \right) du = \int \left( \frac{1}{u^2} - \frac{1}{u^4} \right) du \checkmark
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad & \int \tan^5 x \, dx = \int \frac{\sin^5 x}{\cos^5 x} \, dx \\
 &= \int \frac{\sin^4 x \cdot (-du)}{u^5} \\
 &= \int \frac{(1-u^2)^2 (-du)}{u^5} \rightarrow
 \end{aligned}$$

$$\text{(iv)} \quad \int \tan^3 x \sec^2 x \, dx$$

↖ Done previously!

## Chapter 7: Techniques of Integration

- Integration by parts
- Trigonometric Integrals
- Trigonometric Substitution
- Integration with Partial Fractions
- Improper Integrals

### Trigonometric Substitution

For integrals involving square roots, such as

$$\sqrt{a^2 - x^2}, \quad \sqrt{a^2 + x^2}, \quad \sqrt{x^2 - a^2}$$

strategy for integration is to use :

- Substitutions:
  - ▶  $\sqrt{a^2 - x^2} \rightarrow x = a \sin u$
  - ▶  $\sqrt{a^2 + x^2} \rightarrow x = a \tan u$
  - ▶  $\sqrt{x^2 - a^2} \rightarrow x = a \sec u$
- Useful Trig. Identities:
  - ▶  $\sin^2 u + \cos^2 u = 1$
  - ▶  $1 + \tan^2 u = \sec^2 u$
- Derivatives of trigonometric functions

# Sine Substitution

Ex: Evaluate

(i)  $\int \frac{\sqrt{1-x^2}}{a^2-x^2} dx$

$a^2=1$

$x = a \sin \theta$

$x = 1 \cdot \sin \theta$

$dx = \cos \theta d\theta$

$\sqrt{1-x^2} = \sqrt{1-\sin^2 \theta}$

$= \cos^2 \theta$

$\sqrt{1-x^2} = \cos \theta$

$= \int \cos \theta \cdot \cos \theta d\theta$

$= \int \cos^2 \theta d\theta$

$= \int \frac{1+\cos 2\theta}{2} d\theta$

$= \frac{1}{2} \int d\theta + \frac{1}{2} \int \cos 2\theta d\theta$

$= \frac{1}{2} \theta + \frac{1}{2} \frac{\sin 2\theta}{2} + C$

$= \frac{1}{2} \sin^{-1}(x) + \frac{1}{4} \cdot 2 \sin \theta \cdot \cos \theta + C$

$= \frac{1}{2} \sin^{-1}(x) + \frac{1}{2} \cdot x \cdot \sqrt{1-x^2} + C //$

(ii)  $\int \frac{x^2}{\sqrt{1-x^2}} dx$

$x = \sin \theta$

$\sqrt{1-x^2} = \cos \theta$

$dx = \cos \theta d\theta$

$= \int \frac{\sin^2 \theta \cdot \cos \theta d\theta}{\cos \theta}$

$= \int \sin^2 \theta d\theta$

$= \int \frac{1-\cos 2\theta}{2} d\theta$

$= \frac{1}{2} \int d\theta - \frac{1}{2} \int \cos 2\theta d\theta$

$= \frac{1}{2} \theta - \frac{1}{2} \frac{\sin 2\theta}{2} + C$

$= \frac{1}{2} \theta - \frac{1}{4} \cdot 2 \sin \theta \cdot \cos \theta + C$

$= \frac{1}{2} \sin^{-1}(x) - \frac{1}{2} x \sqrt{1-x^2} + C //$

# Tangent Substitution

Ex: Evaluate

(i)  $\int \frac{\sqrt{1+x^2}}{a^2+x^2} dx$

$a=1$

$x = a \tan \theta$

$x = \tan \theta$

$dx = \sec^2 \theta d\theta$

$\sqrt{1+x^2} = \sqrt{1+\tan^2 \theta}$

$= \sec \theta$

$\sqrt{1+x^2} = \sec \theta$

$= \int \sec \theta \cdot \sec^2 \theta d\theta$

$= \int \sec^3 \theta d\theta$

(Special case Trig. Integrals)

$= \frac{1}{2} \ln |\sec \theta + \tan \theta| + \frac{1}{2} \sec \theta \cdot \tan \theta + C$

$= \frac{1}{2} \ln |x + \sqrt{1+x^2}| + \frac{1}{2} x \sqrt{1+x^2} + C //$

$\frac{1}{9} \frac{\sqrt{9+x^2}}{x} + C$

(ii)  $\int \frac{dx}{x^2 \sqrt{9+x^2}}$

$a=3$

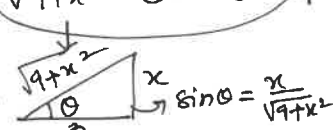
$x = 3 \tan \theta$

$dx = 3 \sec^2 \theta d\theta$

$\sqrt{9+x^2} = \sqrt{9+9 \tan^2 \theta}$

$= 3 \sec \theta$

$\sqrt{9+x^2} = 3 \sec \theta$



$\frac{1}{9 \sin \theta} + C$

$= \int \frac{3 \sec^2 \theta d\theta}{9 \tan^2 \theta \cdot 3 \sec \theta}$

$= \frac{1}{9} \int \frac{1}{\tan^2 \theta} \cdot \sec \theta d\theta$

$= \frac{1}{9} \int \frac{\cos^2 \theta}{\sin^2 \theta} \cdot \frac{1}{\cos \theta} d\theta$

$= \frac{1}{9} \int \frac{\cos \theta d\theta}{\sin^2 \theta}$

$= \frac{1}{9} \int \frac{du}{u^2}$

$= \frac{1}{9} \left( -\frac{1}{u} \right) + C$

$= -\frac{1}{9u} + C$

## Secant Substitution

Ex: Evaluate  $\int \frac{\sqrt{x^2-1} dx}{x^2-a^2} = \int \tan \theta \cdot \sec \theta \tan \theta d\theta$

$$a=1$$

$$\downarrow$$

$$x = a \sec \theta$$

$$x = \sec \theta$$

$$dx = \sec \theta \tan \theta d\theta$$

$$\sqrt{x^2-1} = \sqrt{\sec^2 \theta - 1}$$

$$= \sqrt{\tan^2 \theta}$$

$$\sqrt{x^2-1} = \tan \theta$$

$$= \int \sec \theta \cdot \tan^2 \theta d\theta$$

$$= \int \sec \theta (\sec^2 \theta - 1) d\theta$$

$$= \int \sec^3 \theta d\theta - \int \sec \theta d\theta$$

$$= \frac{1}{2} \ln |\sec \theta + \tan \theta| + \frac{1}{2} \sec \theta \tan \theta - \ln |\sec \theta + \tan \theta| + C$$

$$= \frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta| + C$$

$$= \frac{1}{2} \cdot x \cdot \sqrt{x^2-1} - \frac{1}{2} \ln |x + \sqrt{x^2-1}| + C //$$

## Secant Substitution

Ex: Evaluate  $\int \frac{dx}{\sqrt{x^2-a^2}}$ , where  $a > 0$ .  $\rightarrow \ln a$  should be defined //

$$x = a \sec \theta$$

$$dx = a \sec \theta \tan \theta d\theta$$

$$\sqrt{x^2-a^2} = \sqrt{a^2 \sec^2 \theta - a^2}$$

$$= \sqrt{a^2 (\sec^2 \theta - 1)}$$

$$= \sqrt{a^2 \tan^2 \theta}$$

$$\sqrt{x^2-a^2} = a \tan \theta$$

$$\int \frac{a \sec \theta \tan \theta d\theta}{a \tan \theta}$$

$$= \int \sec \theta d\theta$$

$$= \ln |\sec \theta + \tan \theta| + C$$

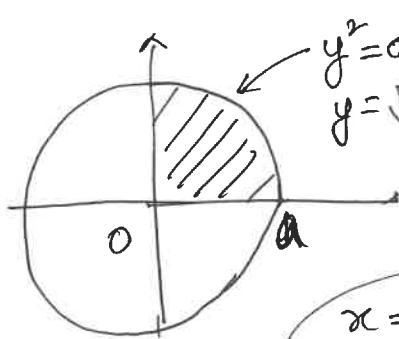
$$= \ln \left| \frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a} \right| + C$$

$$= \ln |x + \sqrt{x^2-a^2}| - \ln |a| + C$$

$$= \ln |x + \sqrt{x^2-a^2}| + C //$$

# Quarter Circle Example $a=1$ centered at the origin.

Ex: Compute the area of the unit circle lying in the first quadrant.



$$\text{Area} = \int_0^a \sqrt{a^2 - x^2} dx$$

$$= \int_0^{\pi/2} a \cos \theta \cdot a \cos \theta d\theta$$

$$= a^2 \int_0^{\pi/2} \cos^2 \theta d\theta$$

previous slides

$$= a^2 \left[ \frac{\theta}{2} + \frac{\sin \theta \cos \theta}{2} \right]_0^{\pi/2}$$

$$= a^2 \left[ \frac{\pi}{4} + \frac{1}{2} \sin\left(\frac{\pi}{2}\right) \cdot \cos\left(\frac{\pi}{2}\right) - 0 - \frac{1}{2} \sin(0) \cos(0) \right]$$

$$= a^2 \cdot \frac{\pi}{4} = \frac{\pi a^2}{4}$$

$x = a \sin \theta$   
 $dx = a \cos \theta d\theta$   
 $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta}$   
 $= \sqrt{a^2 \cos^2 \theta}$   
 $\sqrt{a^2 - x^2} = a \cos \theta$   
 $x=0: \theta=0$   
 $x=a: \sin \theta = 1$   
 $\theta = \pi/2$

## Practice Examples

for a unit circle,  $a=1 \Rightarrow \text{Area} = \frac{\pi}{4}$  in Quad I

Ex: Evaluate

(i)  $\int \sqrt{4-x^2} dx$

$a=2$   
 $x=2 \sin \theta$   
 $dx=2 \cos \theta d\theta$   
 $\sqrt{4-x^2} = \sqrt{4 \cos^2 \theta}$   
 $= 2 \cos \theta$

$= \int 2 \cos \theta \cdot 2 \cos \theta d\theta$   
 $= 4 \int \cos^2 \theta d\theta$   
 Previous slides  
 $= 4 \left( \frac{\theta}{2} + \frac{1}{2} \sin \theta \cos \theta \right)$   
 $= 2\theta + 2 \sin \theta \cos \theta + C$   
 $= 2 \sin^{-1} \left( \frac{x}{2} \right) + 2 \cdot \frac{x}{2} \cdot \frac{\sqrt{4-x^2}}{2} + C$   
 $= 2 \sin^{-1} \left( \frac{x}{2} \right) + \frac{1}{2} x \sqrt{4-x^2} + C$

(ii)  $\int \sqrt{x^2-9} dx$

$a=3$   
 $x=3 \sec \theta$   
 $dx=3 \sec \theta \tan \theta d\theta$   
 $\sqrt{x^2-9} = \sqrt{9 \sec^2 \theta - 9}$   
 $= \sqrt{9(\sec^2 \theta - 1)}$   
 $= 3 \tan \theta$   
 $\sqrt{x^2-9} = 3 \tan \theta$   
 $\frac{\sqrt{x^2-9}}{3} = \tan \theta$   
 $\sec \theta = \frac{x}{3}$

$= \int 3 \tan \theta \cdot 3 \sec \theta \cdot \tan \theta d\theta$   
 $= 9 \int \sec \theta \cdot \tan^2 \theta d\theta$   
 $= 9 \int \sec \theta \cdot (\sec^2 \theta - 1) d\theta$   
 $= 9 \int \sec^3 \theta d\theta - 9 \int \sec \theta d\theta$   
 $= 9 \left[ \frac{1}{2} \ln |\sec \theta + \tan \theta| + \frac{1}{2} \sec \theta \cdot \tan \theta - \ln |\sec \theta + \tan \theta| \right] + C$   
 $= \frac{9}{2} \sec \theta \tan \theta - \frac{9}{2} \ln |\sec \theta + \tan \theta| + C$   
 $= \frac{9}{2} \sqrt{x^2-9} - \frac{9}{2} \ln \left| \frac{x}{3} + \frac{\sqrt{x^2-9}}{3} \right| + C$

# Practice Examples

Ex: Evaluate

(i)  $\int \frac{\sqrt{25+x^2} dx}{a^2+x^2}$

$a=5$   
 $x=5 \tan \theta$   
 $dx=5 \sec^2 \theta d\theta$   
 $\sqrt{25+x^2}=5 \sec \theta$

$= \int 5 \sec \theta \cdot 5 \sec^2 \theta d\theta$

$= 25 \int \sec^3 \theta d\theta$

$= 25 \left[ \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right] + C$

$= \frac{25}{2} \sec \theta \tan \theta + \frac{25}{2} \ln |\sec \theta + \tan \theta| + C$

$= \frac{25}{2} \cdot \frac{\sqrt{25+x^2}}{5} \cdot \frac{x}{5} + \frac{25}{2} \ln \left| \frac{\sqrt{25+x^2} + x}{5} \right| + C$

$= \frac{x}{2} \sqrt{25+x^2} + \frac{25}{2} \ln |x + \sqrt{25+x^2}| + C //$

(ii)  $\int \frac{1}{(x^2+4)^{3/2}} dx = \int \frac{2 \sec^2 \theta d\theta}{8 \sec^3 \theta}$

$x=2 \tan \theta$   
 $dx=2 \sec^2 \theta d\theta$   
 $(x^2+4)^{3/2} = (4 \sec^2 \theta)^{3/2}$   
 $= 4^{3/2} (\sec^2 \theta)^{3/2}$   
 $= 8 \sec^3 \theta$

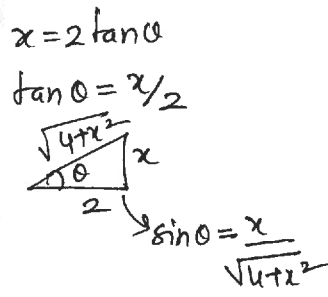
$= \frac{1}{4} \int \frac{1}{\sec \theta} d\theta$

$= \frac{1}{4} \int \cos \theta d\theta$

$= \frac{1}{4} \sin \theta + C$

$= \frac{1}{4} \cdot \frac{x}{\sqrt{4+x^2}} + C$

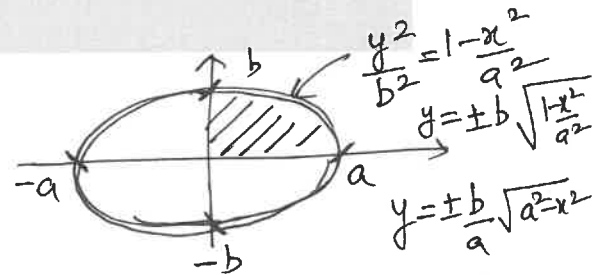
$= \frac{x}{4\sqrt{4+x^2}} + C //$



## Example

Ex: Find the area enclosed by the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Area enclosed by the ellipse

$= 4 \times \text{Area in Quadrant I}$

$= 4 \int_0^a \frac{b}{a} \sqrt{a^2-x^2} dx$

$= \frac{4b}{a} \int_0^a \sqrt{a^2-x^2} dx$

Unit circle calculation  
 Previous slide.

$\frac{4b}{a} \cdot \frac{\pi a^2}{4} = \pi ab //$

# Practice Examples

Ex: Evaluate

(i)  $\int_0^{5/2} \frac{dx}{\sqrt{25-x^2}}$

$a=5$   
 $x=5\sin\theta$   
 $dx=5\cos\theta d\theta$   
 $\sqrt{25-x^2}=5\cos\theta$

$x=0: \theta=0$   
 $x=5/2: \frac{5}{2}=5\sin\theta$   
 $\frac{1}{2}=\sin\theta$   
 $\theta=\pi/6$

$\int_0^{\pi/6} \frac{5\cos\theta d\theta}{5\cos\theta}$   
 $= \int_0^{\pi/6} d\theta$   
 $= |\theta|_0^{\pi/6}$   
 $= \frac{\pi}{6} - 0$   
 $= \pi/6 //$

Do in Class

Practice Example

$\int \frac{\sqrt{x^2-2x-3}}{x-1} dx$

(ii)  $\int \frac{x}{\sqrt{3-2x-x^2}} dx$

$3-2x-x^2 = 3 - (x^2+2x)$   
 $= 3 - (x^2+2x+1-1)$   
 $= 3 - [(x+1)^2-1]$   
 $= 3 - (x+1)^2 + 1$   
 $= 4 - (x+1)^2$

$u=x+1 \rightarrow x=u-1 \rightarrow dx=du$   
 $\sqrt{3-2x-x^2} = \sqrt{4-(x+1)^2} = \sqrt{4-u^2}$

$\int \frac{x dx}{\sqrt{3-2x-x^2}} = \int \frac{(u-1) du}{\sqrt{4-u^2}}$   
 $= \int \frac{u du}{\sqrt{4-u^2}} - \int \frac{du}{\sqrt{4-u^2}}$   
 (I) (II)

$= -\sqrt{3-2x-x^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C$

(I)  $\rightarrow \int \frac{u du}{\sqrt{4-u^2}} = \int \frac{2\sin\theta \cdot 2\cos\theta d\theta}{2\cos\theta}$

$a=2$   
 $u=2\sin\theta$   
 $\sqrt{4-u^2}=2\cos\theta$   
 $du=2\cos\theta d\theta$

$= 2 \int \sin\theta d\theta$   
 $= 2(-\cos\theta)$   
 $= -2\cos\theta$   
 $= -2 \cdot \frac{\sqrt{4-u^2}}{2}$   
 $= -\sqrt{4-u^2}$   
 $= -\sqrt{3-2x-x^2} //$

(II)  $\int \frac{du}{\sqrt{4-u^2}} = \int \frac{2\cos\theta d\theta}{2\cos\theta} = \int d\theta = \theta = \sin^{-1}\left(\frac{u}{2}\right) = \sin^{-1}\left(\frac{x+1}{2}\right) //$