Chapter 7: Techniques of Integration

- Integration by parts
- Trigonometric Integrals
- Trigonometric Substitution
- Integration with Partial Fractions
- Improper Integrals

Integration by Parts

Integration by parts is the name of the formula we get if we undo the Product Rule.

$$\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

$$(u \cdot v)' = u \cdot v' + v \cdot u'$$

$y = f(x)$

$dy = \frac{df}{dx} \cdot dx$

$\begin{cases} dv = \frac{dv}{dx} \cdot dx \\ du = \frac{du}{dx} dx. \end{cases}$

Integrating both sides,

$$\int (u \cdot v)' dx = \int u \cdot v' dx + \int v \cdot u' dx$$

FTC:

$$(u \cdot v) = \int u \cdot dv + \int v \cdot du$$

Integration by Parts:

$$\boxed{\int u \cdot dv = u \cdot v - \int v \cdot du}$$

In this form we take the integral of one product and express it in terms of the integral of a different product. If we express it like that, it does not seem too useful. However, if the second integral is easier, then this process helps us.

Examples $\quad \int u\cdot dv = u\cdot v - \int v\cdot du$

Ex: Evaluate the integrals:

(i) $\int xe^x\,dx = xe^x - \int e^x\,dx$

$u = x \quad dv = e^x\,dx$

$du = 1.dx \quad V = e^x$

$= xe^x - e^x + C$

(ii) $\int x\cos x\,dx = x\sin x - \int \sin x\,dx$

$u = x \quad dV = \cos x\,dx$

$du = 1.dx \quad V = \sin x$

$= x.\sin x - (-\cos x) + C$

$= x\sin x + \cos x + C$ //

(iii) $\int x^2.e^x dx = x^2.e^x - \int 2xe^x dx$

$u = x^2 \quad dv = e^x\,dx$

$\downarrow \quad \downarrow$

$du = 2x\,dx \quad V = e^x$

$= x^2e^x - 2\int xe^x dx$

$= x^2e^x - 2\left[xe^x - e^x\right] + C$

$= x^2e^x - 2xe^x + 2e^x + C$ //

(iv) $\int x\sin x\,dx = -x\cos x + \int(-\cos x)\,dx$

$u = x \quad dv = \sin x\,dx$

$\downarrow \quad \downarrow$

$du = 1.dx \quad V = -\cos x$

$= -x\cos x - \int \cos x\,dx$

$= -x\cos x - \sin x + C$ //

Examples

Ex: Evaluate the integrals:

(i) $\int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^{\cancel{2}}}{2}\cdot\frac{1}{\cancel{x}}\,dx$

$u = \ln x$	$dV = x\,dx$
$du = \frac{1}{x}\,dx$	$V = \frac{x^2}{2}$

$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$

$= \frac{x^2}{2}\ln x - \frac{1}{2}\cdot\frac{x^2}{2} + C$

$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$ //

(ii) $\int x^3e^x\,dx = x^3e^x - \int e^x.3x^2\,dx$

$u = x^3 \quad dv = e^x\,dx$

$du = 3x^2dx \quad V = e^x$

$= x^3e^x - 3\int x^2e^x\,dx$

$= x^3e^x - 3\left[x^2e^x - 2xe^x + 2e^x\right] + C$

$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$ //

Basic Functions

Ex: Evaluate the integrals:

(i) $\int \ln x\, dx = \int 1 \cdot \ln x\, dx$

$u = \ln x \quad dv = 1 \cdot dx$

$du = \frac{1}{x} dx \quad v = x$

$= x \cdot \ln x - \int x \cdot \frac{1}{x} dx$

$= x \cdot \ln x - \int 1 \cdot dx$

$= x \ln x - x + C$ //

(ii) $\int \arctan(x)\, dx = \int 1 \cdot \tan^{-1}(x)\, dx$

$u = \tan^{-1}(x) \quad dv = 1\, dx$

$du = \frac{1}{1+x^2} dx \quad v = x$

$= x \cdot \tan^{-1} x - \int x \cdot \frac{1}{1+x^2} dx$

$= x \tan^{-1} x - \int \frac{x}{1+x^2} dx$

$t = 1 + x^2$

$dt = 2x\, dx$

$\frac{dt}{2} = x\, dx$

$= x \tan^{-1} x - \int \frac{1}{t} \cdot \frac{dt}{2}$

$= x \tan^{-1} x - \frac{1}{2} \int \frac{1}{t} dt$

$= x \tan^{-1} x - \frac{1}{2} \ln|t| + C$

$= x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$ //

Integration by Parts and Substitution Rule

Ex: Evaluate the integrals:

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}} dx$

$= \frac{1}{2u} dx$

$2u\, du = dx$

(i) $\int e^{\sqrt{x}} dx = \int e^u \cdot 2u\, du$

$= 2 \int u e^u\, du$

$= 2\left[u \cdot e^u - \int e^u du\right]$

$= 2ue^u - 2e^u + C = 2\sqrt{x} e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$

(ii) $\int \sin(\sqrt{x})\, dx$

$= \int \sin(u) \cdot 2u\, du$

$= 2 \int u \cdot \sin u\, du$

$= 2\left[u(-\cos u) - \int (-\cos u) \cdot 1 \cdot du\right]$

$= -2u \cos u + 2 \int \cos u\, du$

$= -2u \cos u + 2 \sin u + C = -2\sqrt{x} \cos(\sqrt{x}) + 2 \sin(\sqrt{x}) + C$ //

(iii) $\int \cos(\ln x)\, dx = \int \cos(u) \cdot x \cdot du$

$u = \ln x \Rightarrow x = e^u$

$du = \frac{1}{x} dx$

$= \int e^u \cos u\, du$

$= e^u \cos u + \int e^u \sin u\, du$

$\downarrow$

?

(iv) $\int \sqrt{x} e^{\sqrt{x}} dx$

$= \int u \cdot e^u \cdot 2u\, du$

$= 2 \int u^2 e^u\, du$

Previous Calculations from last page

$= 2\left[u^2 e^u - 2ue^u + 2e^u\right] + C$

$= 2xe^{\sqrt{x}} - 4\sqrt{x} e^{\sqrt{x}} + 4e^{\sqrt{x}} + C$ //

Definite Integrals

Integration by Parts & Fundamental Theorem of Calculus (Part 2), give us

$$\boxed{\int_a^b f(x)g'\,dx = [f(x)g(x)]_a^b - \int_a^b g(x)f'(x)\,dx}$$

Ex: Evaluate the integrals:

(i) $\int_0^1 xe^x\,dx$

$= \left[xe^x\right]_0^1 - \int_0^1 e^x \cdot 1\,dx$

$= (1\cdot e^1 - 0) - \left|e^x\right|_0^1$

$= e - (e^1 - e^0)$

$= \cancel{e} - \cancel{e} + 1$

$= 1$

(ii) $\int_1^3 x^3 \ln x\,dx$

$= \left|\ln x \cdot \frac{x^4}{4}\right|_1^3 - \int_1^3 \frac{1}{x}\cdot\frac{x^4}{4}\,dx$

$= \frac{1}{4}\left(3^4 \ln 3 - \ln 1\right) - \frac{1}{4}\int_1^3 x^3\,dx$

$= \frac{1}{4}\left(3^4 \ln 3\right) - \frac{1}{4}\left|\frac{x^4}{4}\right|_1^3$

$= \frac{81}{4}\ln 3 - \frac{1}{16}\left(3^4 - 1\right) = \frac{81}{4}\ln 3 - \frac{\cancel{80}^{5}}{\cancel{16}}$

$= \frac{81}{4}\ln 3 - 5$ //

Reduction Formulas

Ex: Prove the reduction formulas:

(i) $\int t^n e^t\,dt = t^n e^t - n\int t^{n-1}e^t\,dt$ ✓

↑ directly from Int. by parts formula.

$dv = e^t dt$

$v = e^t$

$u = t^n$

$du = n\cdot t^{n-1}\,dt$

(ii) $\int \cos^n\theta\,d\theta = \frac{1}{n}\sin\theta\cos^{n-1}\theta + \frac{n-1}{n}\int \cos^{n-2}\theta\,d\theta$

$\int \cos^{n-1}\theta \cdot \cos\theta\,d\theta = \cos^{n-1}\theta \cdot \sin\theta - \int \sin\theta \cdot (n-1)\cos^{n-2}\theta\,d\theta$

$= \downarrow \; - (n-1)\int \cos^{n-2}\theta \cdot \sin\theta\,d\theta$

$= \downarrow \; - (n-1)\Big[\cos^{n-2}\theta(-\cos\theta)$

$+ \int \cos\theta \cdot (n-2)\cos^{n-3}\theta \cdot \sin\theta\,d\theta$

$u = \cos^{n-1}\theta$

$du = (n-1)\cos^{n-2}\theta\,d\theta$

$dv = \cos\theta\,d\theta$

$v = \sin\theta$

Recurring Integrals

Some version of the following integral have this recurring characteristic...

$$\int e^{ax}\sin(bx)\,dx \quad \text{or} \quad \int e^{ax}\cos(bx)\,dx$$

Ex: Evaluate $\int e^x \sin x\,dx$

$$I = \int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx$$

IP1: $u = \sin x$, $dv = e^x dx$; $du = \cos x\,dx$, $v = e^x$

$$= e^x \sin x - \left[e^x \cos x - \int e^x(-\sin x)\,dx\right]$$

IP2: $u = \cos x$, $dv = e^x dx$; $du = -\sin x\,dx$, $v = e^x$

$$I = e^x \sin x - e^x \cos x - I$$

$$2I = e^x(\sin x - \cos x)$$

$$I = \frac{e^x}{2}(\sin x - \cos x)$$

Practice Examples

Ex: Evaluate the integrals

Recurring + Definite

(i) $\int_a^b e^x \cos x\,dx$

$$I = \int_a^b e^x \cos x\,dx$$

$$= \left|e^x \sin x\right|_a^b - \int_a^b e^x \sin x\,dx$$

$$= \left|e^x \sin x\right|_a^b - \left[\left|-e^x \cos x\right|_a^b - \int_a^b e^x(-\cos x)\,dx\right]$$

$$= \left|e^x \sin x + e^x \cos x\right|_a^b - \int_a^b e^x \cos x\,dx$$

$$I = \left|e^x(\sin x + \cos x)\right|_a^b - I$$

$$2I =$$

$$I = \frac{1}{2}\left|e^x(\sin x + \cos x)\right|_a^b \quad //$$

Int. by Parts + Definite

(ii) $\int_0^{\pi} x^2 \sin x\,dx$

$$= \left|-x^2 \cos x\right|_0^{\pi} - \int_0^{\pi} -2x\cos x\,dx$$

$$= \left|-\pi^2 \cos\pi + 0\right| + 2\int_0^{\pi} x\cos x\,dx$$

$$= -\pi^2(-1) + 2\left[\left|x\sin x\right|_0^{\pi} - \int_0^{\pi} \sin x\,dx\right]$$

$$= \pi^2 + 2\left|\pi\sin\pi - 0\right| - 2\left|-\cos x\right|_0^{\pi}$$

$$= \pi^2 + 2\cdot 0 - 2(-\cos\pi + \cos 0)$$

$$= \pi^2 - 2(1+1)$$

$$= \pi^2 - 4 \quad //$$

Practice Examples

Ex: Evaluate the integrals:

(i) $\int_1^2 2te^{3t^2}\,dt$

$x = 3t^2$
$dx = 6t\,dt$
$\frac{dx}{3} = 2t\,dt$

$t=1 : x = 3$
$t=2 : x = 12$

$= \int_3^{12} e^x \cdot \frac{dx}{3}$

$= \frac{1}{3}\int_3^{12} e^x\,dx$

$= \frac{1}{3}\left| e^x \right|_3^{12}$

$= \frac{1}{3}\left(e^{12} - e^3\right)$

(ii) $\int e^{2t}\cos t\,dt$

$I = \int e^{2t}\cos t\,dt = e^{2t}\cdot \sin t - \int 2e^{2t}\sin t\,dt$

$I = e^{2t}\sin t - 2\left[-e^{2t}\cos t + 2\int e^{2t}\cos t\,dt\right]$

$I = e^{2t}\sin t + 2e^{2t}\cos t - 4I$

$5I = e^{2t}\left(\sin t + 2\cos t\right)$

$\boxed{I = \frac{1}{5}e^{2t}\left(\sin t + 2\cos t\right)}$

Practice Examples

Ex: Evaluate the integrals:

(iii) $\int_0^{\pi/2} (1 - \sin^2 t)\sin t\,dt$

$= \int_0^{\pi/2} \cos^2 t \cdot \sin t\,dt$

$u = \cos t$
$du = -\sin t\,dt$

$t = 0 : u = 1$
$t = \pi/2 : u = 0$

$= \int_1^0 u^2 \cdot (-du)$

$= -\int_1^0 u^2\,du$

$= \int_0^1 u^2\,du$

$= \left|\frac{u^3}{3}\right|_0^1 = \frac{1}{3}(1-0) = 1/3$

(iv) $\int \arcsin(x)\,dx = \int 1 \cdot \arcsin(x)\,dx$

$u = \arcsin(x)$
$du = \frac{1}{\sqrt{1-x^2}}\,dx$
$dv = 1\,dx$
$v = x$

$= x \cdot \arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$

$t = \sqrt{1-x^2}$
$dt = \frac{1}{2\sqrt{1-x^2}} \cdot -\cancel{2}x\,dx$
$-dt = \frac{x\,dx}{\sqrt{1-x^2}}$

$= x \cdot \arcsin(x) - \int -dt$

$= x\arcsin(x) + t + C$

$= x\arcsin(x) + \sqrt{1-x^2} + C$