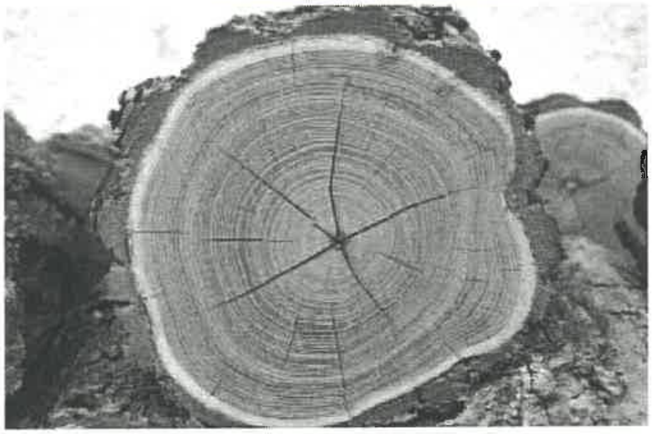
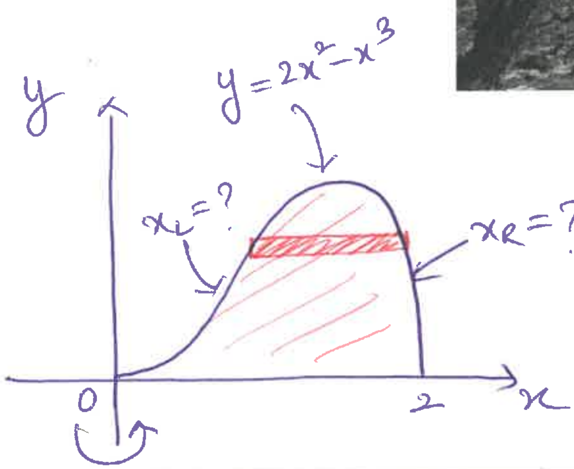
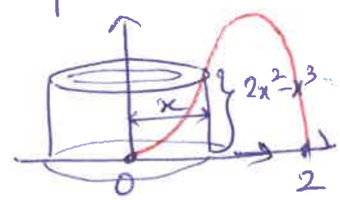
Cylindrical Shells

Sometimes it is easier to set up the volume problem in an alternate way.... → Cylindrical shells.





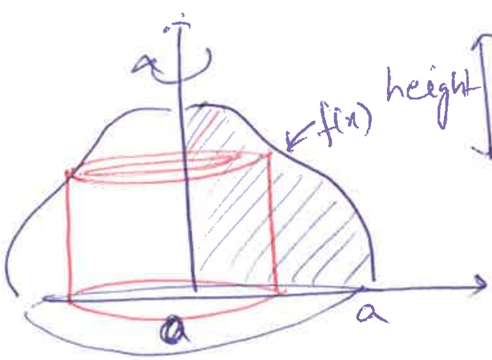
How to determine x_R or x_L to find the width of rectangular strip?

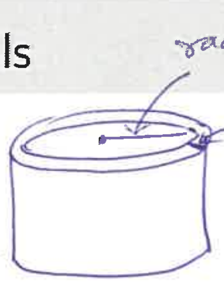


Using cylindrical shell method

$$V=\int_0^2 \underbrace{(2\pi x)}_{\text{Circumference}}\ \underbrace{(2x^2-x^3)}_{\text{height}}\ \underbrace{dx}_{\text{thickness}}$$

Method of Shells





a = inner rad

b = outer radius

$\boxed{b=a+dx}$

Volume of cylindrical shell

$= \underbrace{2\pi\,(\text{radius})}_{\text{Circumference}}(\text{height})(\text{thickness})$

$V_{\text{outer shell}} - V_{\text{inner shell}}$

$=\pi b^2 h - \pi a^2 h$

$=\pi(a+dx)^2 h - \pi a^2 h$

$=\pi(a^2+dx^2+2a\,dx)h-\pi a^2 h$

$=\cancel{\pi a^2 h}+\pi dx^2 h+2\pi a\,dx\,h$
$\quad -\cancel{\pi a^2 h}$

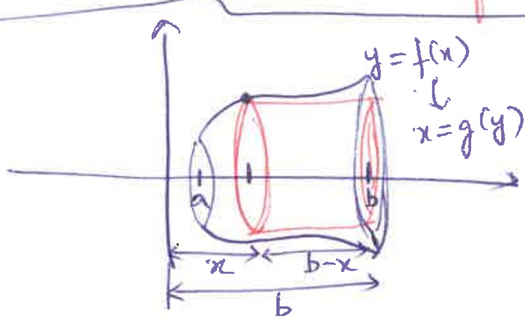
$=2\pi a h\,dx+\underbrace{\pi h(dx)^2}_{\text{very small as } dx\to 0}$

$=2\pi a h dx + 0$

$=2\pi a h dx.$

Using cylindrical shells,

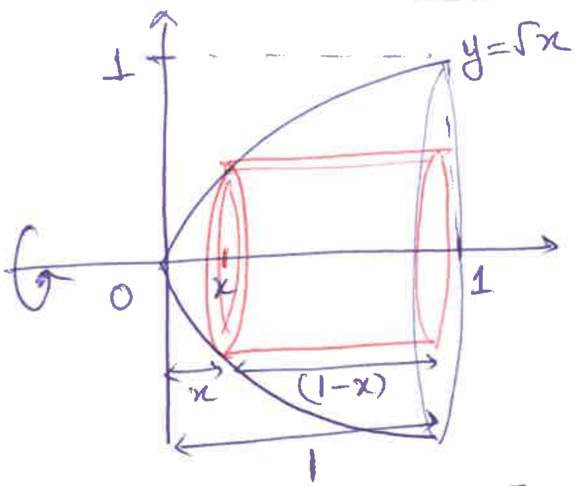
$$\text{Volume} = \int_a^b \underbrace{(2\pi x)}_{\text{Circumference}}\ \underbrace{(f(x))}_{\text{height}}\ \underbrace{dx}_{\text{thickness}}$$



$$\text{Volume}=\int_a^b (2\pi y)(b-g(y))\,dy$$

Example

Ex: Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.



$y = \sqrt{x}$

$y^2 = x$

$\boxed{1-x = 1-y^2}$

$$\text{Volume} = \int_0^1 (2\pi y)(1-x)\,dy$$

$$= \int_0^1 2\pi y(1-y^2)\,dy$$

$$= 2\pi \int_0^1 (y-y^3)\,dy$$

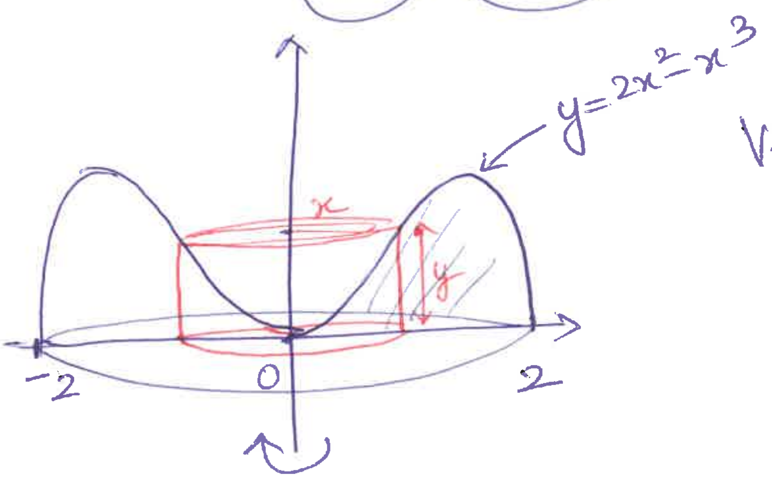
$$= 2\pi \left| \frac{y^2}{2} - \frac{y^4}{4} \right|_0^1$$

$$= 2\pi\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \frac{\pi}{2} \;//$$

Example

Ex: Use cylindrical shells to find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$ between $x = 0$ and $x = 2$.



$$\text{Volume} = \int_0^2 (2\pi x)\cdot y \cdot dx$$

$$= \int_0^2 2\pi x \cdot (2x^2 - x^3)\,dx$$

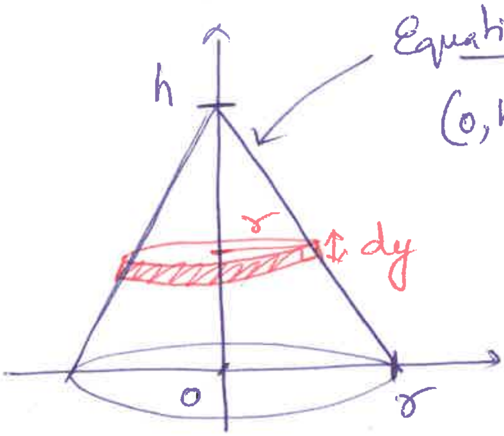
$$= 2\pi \int_0^2 (2x^3 - x^4)\,dx$$

$$= 2\pi \left| \frac{\cancel{2} \cdot x^4}{\cancel{2}\cancel{4}} - \frac{x^5}{5} \right|_0^2$$

$$= 2\pi\left(\frac{2^4}{2} - \frac{2^5}{5}\right) = \frac{16}{5}\pi \;//$$

Practice Example

Ex: The volume of a cone with a height h and base radius r is $V = \frac{1}{3}\pi r^2 h$. Derive this formula with an integral using the disk method.



Equation of line: $(0,h)$ & $(r,0)$

$m = \frac{h-0}{0-r} = -\frac{h}{r}$

$y - 0 = -\frac{h}{r}(x - r)$

$\boxed{y = -\frac{h}{r}x + h} \Rightarrow \frac{xh}{r} = h - y$

$x = \frac{r}{h}(h-y)$

$\boxed{x = r - \frac{r}{h}y}$

$V_{disk} = \pi r^2\, dy$
$= \pi x^2\, dy$
$= \pi\left(r - \frac{r}{h}y\right)^2 dy$

$\Rightarrow$ Volume of Solid $= \int_0^h \pi\left(r - \frac{r}{h}y\right)^2 dy$

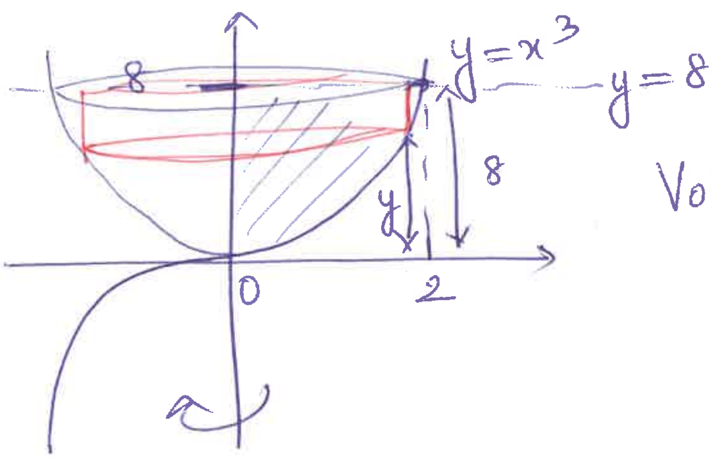
$= \pi \int_0^h \left(r^2 + \frac{r^2}{h^2}y^2 - \frac{2r^2}{h}y\right) dy$

$= \pi \left| r^2 y + \frac{r^2}{h^2}\cdot\frac{y^3}{3} - \frac{2r^2}{h}\cdot\frac{y^2}{2} \right|_0^h$

$= \pi\left(r^2 h + \frac{r^2}{3h^2}\cdot h^3 - r^2 h\right) = \frac{\pi r^2 h}{3}$ //

Practice Example

Ex: An area is bounded by $y = x^3$, $y = 8$, and $x = 0$. Find the volume generated by rotating the area around the $y-$axis.



Shell method!

Volume $= \int_0^2 (2\pi x)(8 - y)\, dx$

$= 2\pi \int_0^2 x(8 - x^3)\, dx$

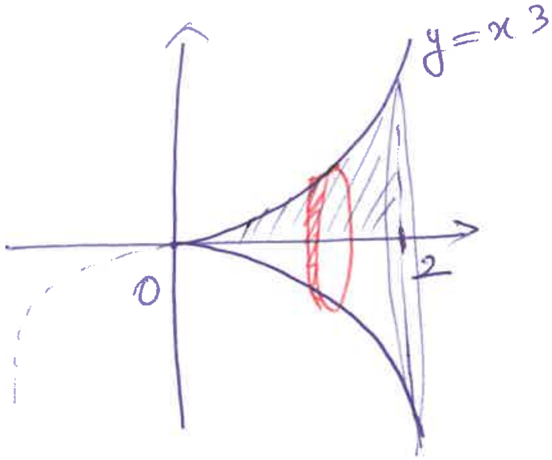
$= 2\pi \left| 8\frac{x^2}{2} - \frac{x^5}{5} \right|_0^2$

$= 2\pi \left| 4\cdot 2^2 - \frac{2^5}{5} \right|$

$= \frac{96\pi}{5}$ //

Practice Example

$\boxed{\text{Ex:}}$ An area is bounded by $y = x^3$, ~~$y = 8$~~ $x=2$, and ~~$x = 0$~~ $y=0$. Find the volume generated by rotating the area around the ~~y~~ x–axis.



Disk method:

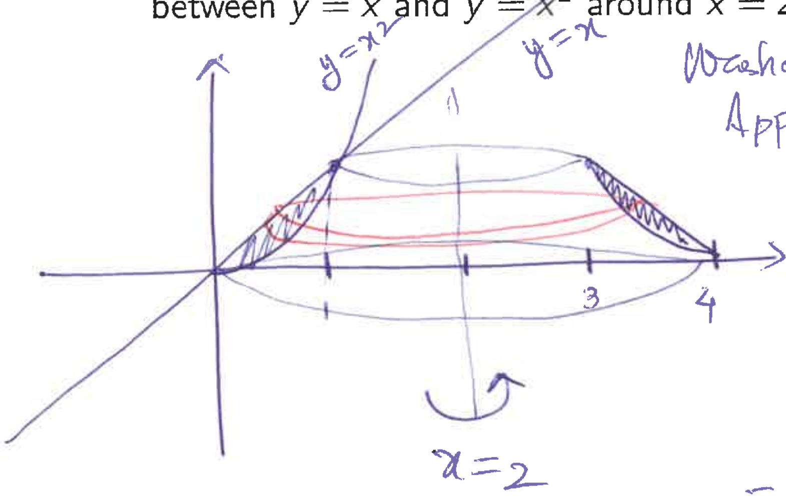
$$V_{disk} = \pi y^2 \, dx = \pi x^6 \, dx$$

Volume of solid

$$= \int_0^2 \pi x^6 \, dx = \pi \left[\frac{x^7}{7}\right]_0^2 = \frac{\pi}{7} \cdot 2^7 = \frac{128\pi}{7}$$

Practice Example

$\boxed{\text{Ex:}}$ Find the volume of a solid generated by rotating the region bounded between $y = x$ and $y = x^2$ around $x = 2$



Washer Approach

$y = x \rightarrow r_{out} = 2 - x = 2 - y$

$y = x^2 \rightarrow r_{in} = 2 - x = 2 - \sqrt{y}$

$x = \sqrt{y}$

Volume of solid

$$= \int_0^1 \left[\pi (2-y)^2 - \pi (2-\sqrt{y})^2\right] dy$$

or, Cylindrical Shell Approach.

Refer: Textbook Sec 6.3 example 5

Work

In physics, the work done by applying a **constant force** is defined as the product of force and distance. Work $= Fd =$ Force $\times$ dist.

The force needed to overcome gravity and lift an object is called the weight of the object.

$$F = mg \quad \text{or} \quad F = \text{mass} \times \text{acceleration}.$$

where $g = 9.8$ m/s^2 is the acceleration due to gravity and is a constant.

Ex: Determine the mass of the cylinder with radius 2m, height 3m, and a density 500kg/m^3. Determine the work done in lifting the object up 10m.

Mass = Density $\times$ Volume

Volume of cylinder $= \pi r^2 h$
$= \pi \cdot 2^2 \cdot 3$
$= 12\pi$

Mass = Density $\times$ Volume $= 500 \times 12\pi = 6000\pi$ kg.

Work done $= F \times d$
$= mg \times d$
$= (6000\pi \times 9.8 \times 10)$ Nm.

Units: Force $\rightarrow$ N ($\uparrow$ kgm/s^2)
Work $\rightarrow$ Nm
Vol. $\rightarrow$ cm^3/m^3
Mass $\rightarrow$ kg/g/...

Variable Force

If instead of a constant force, we have a variable force $F(x)$, then the work done by the force in moving the object from point $x = a$ to $x = b$ is given by

$[a,b] \rightarrow \Delta x \rightarrow x_i^*$ in $[x_{i-1}, x_i]$
$f(x_i^*) \cdot \Delta x \rightarrow$

$$\boxed{W = \int_a^b F(x)\,dx} \xleftarrow{n \to \infty}$$

$W_i \approx f(x_i^*) \cdot \Delta x$
$W \approx \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x$

Ex: Suppose the force acting on a particle at position $'x'$ is given as a function

$$f(x) = 3x^2 + x.$$

What is the work required to move the object from position $x = 1$ to position $x = 3$?

Work done $= \int_1^3 f(x)\,dx = \int_1^3 (3x^2 + x)\,dx$
$= \left| \cancel{3} \cdot \frac{x^3}{\cancel{3}} + \frac{x^2}{2} \right|_1^3$
$= \left(3^3 + \frac{3^2}{2}\right) - \left(1 + \frac{1}{2}\right) = 30$

Hooke's Law - Spring Force

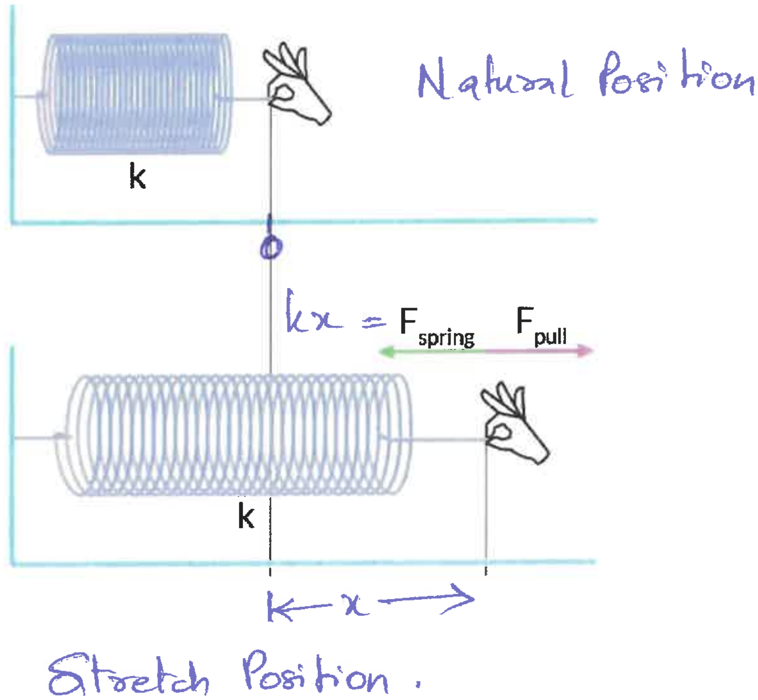
A common example of variable force is that required to stretch a spring. In the simplest scenario the force required to stretch the spring is proportional to the length the spring has already been stretched.

Hooke's law:

Force required $\propto x$

$$F = kx$$

where 'k' is a positive constant called the spring constant.

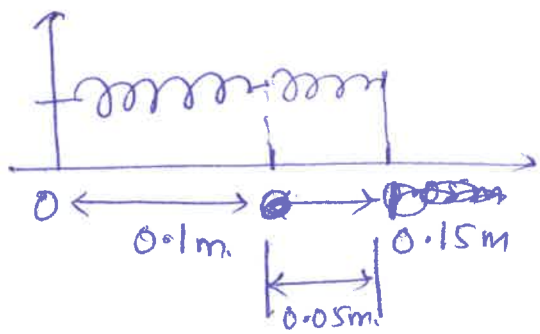


Example

What its spring constant?

Ex: Suppose a 40 N force is required to hold a spring at 0.05 m past its natural length of 0.1 m. How much work was required to stretch the spring from its natural length to the current length?

According to Hooke's law,



$$F = k \cdot x$$

$$40 = k(0.05)$$

$$k = \frac{40}{0.05} = \frac{40 \times 100}{5} = 800 \text{ N/m.}$$

$$\text{work done.} = F.d$$

$$= 40(0.05)$$

$$= 2 \text{ Nm.}$$

Lifting of Cable Example

Ex: A 200 lb cable 100ft long and hangs vertically from the top of a tall building.

(a) How much work is required to lift the cable to the top of the building?

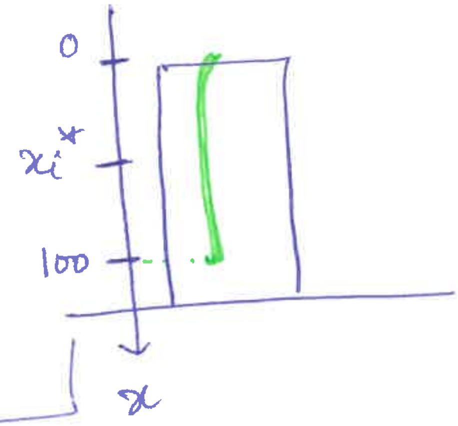
(b) How much work is required to pull up only 20 feet of the cable?

(a) Mass of per ft of cable $= \frac{200}{100} = 2$ lb/ft.

Weight of i^{th} part $= 2 \cdot \Delta x = 2\Delta x$ lb

Work done on i^{th} part

$$= (2\Delta x) \cdot x_i = 2 x_i^* \Delta x$$



Mass $m = 200$ Lb
Length $l = 100$ ft.

Work done

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} 2x_i^* \Delta x = \int_0^{100} 2x\,dx = \frac{2 \cdot x^2}{2}\Big|_0^{100} = 10,000 \text{ ft-lb}$$

Lifting of Cable Example

If the origin was at the bottom of the building,

$$W = \int_0^{100} 2(100 - x)\,dx = 10000 \text{ ft-lb}$$

P

(b) Work done to pull only the top 20 ft of the cable

$$W_1 = \int_0^{20} 2x\,dx = x^2\Big]_0^{20} = 400 \text{ ft-Lb}$$

Every part of the lower 80 ft cable also moves the same distance (20 ft $\uparrow$), so

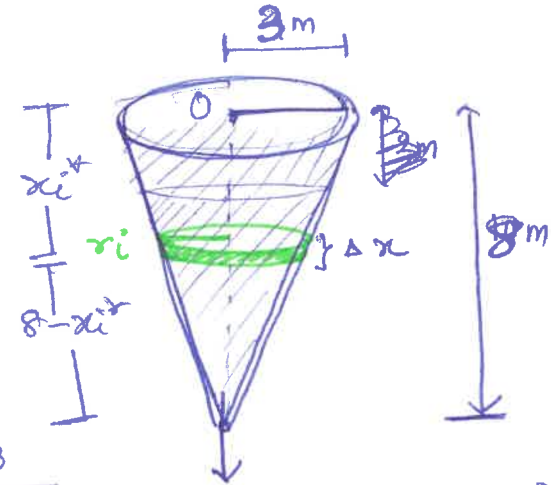
$$W_2 = \int_{20}^{100} \underbrace{20}_{\text{distance}} \cdot \underbrace{2\,dx}_{\text{force}} = 3200 \text{ ft-lb}.$$

$\therefore$ Total work done $= W_1 + W_2 = 400 + 3200 = 3600$ ft-lb //

Tank Example

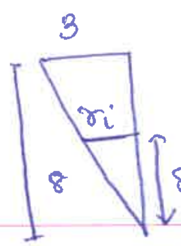
$\boxed{\text{Ex:}}$ A tank is in the shape of an inverted cone (the point is at the bottom) with radius 3 m, height 8 m. It is filled with a fluid that has density 1100 kg/m^3. What is the work required to pump all the fluid to the top of the tank?

$r_i = \frac{3}{8}(8-x_i)$



Volume of the ith layer,

$$V_i \approx \pi r_i^2 \Delta x$$
$$= \pi \cdot \frac{9}{64}(8-x_i)^2 \Delta x$$



Density, $\rho = 1100\,kg/m^3$

From similar triangles,

$$\boxed{\frac{r_i}{8-x_i} = \frac{3}{8}}$$

Mass of ith layer

$$m_i = \text{density} \times \text{volume}$$
$$\simeq 1100 \times \frac{9\pi}{64}(8-x_i)^2 \Delta x$$

Force required, $F_i = m_i g$

Tank Example

$$\simeq \frac{9900}{64}\pi (8-x_i)^2 \Delta x \times 9.8$$
$$= \alpha (8-x_i)^2 \Delta x \quad \text{where } \alpha = \frac{9900 \times 9.8 \times \pi}{64}$$

Work done, $W_i \approx F \cdot d$

$$\simeq F_i x_i$$
$$= \alpha (8-x_i)^2 \cdot x_i \Delta x$$

$\therefore$ Total Work done, $W = \lim_{n\to\infty} \sum_{i=1}^{n} W_i = \int_0^8 \alpha (8-x)^2 \cdot x \cdot dx$

$$= \alpha \int_0^8 x(64 + x^2 - 16x)\,dx$$
$$= \alpha \left| 64 \cdot \frac{x^2}{2} + \frac{x^4}{4} - 16 \cdot \frac{x^3}{3} \right|_0^8$$

Practice Example - Spring

Ex: Suppose a spring extends 5 cm past its natural length when one kilogram is hung from its end. How much work is done to extend the spring from 5 cm past its natural length to 7 cm past its natural length?

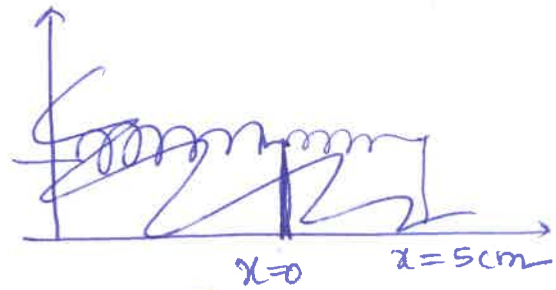
$F = kx$

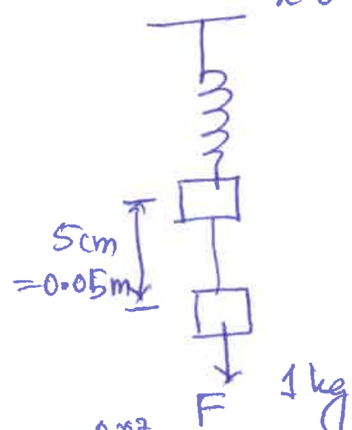
$9.8 = k \times 0.05$

$k = \dfrac{9.8}{0.05}$ N/m.

When $x = 7\text{ cm} = 0.07$ m

$F = k \cdot x$





$M = 1$ kg

$F = m \cdot g$

$= 9.8$ N

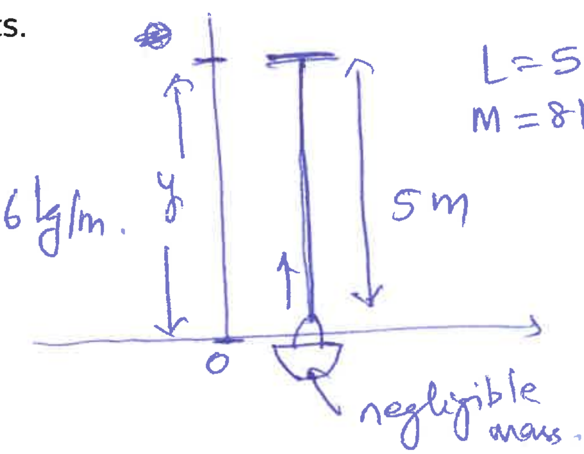
Work done $= \displaystyle\int_{0.05}^{0.07} F\,dx$

$= \dfrac{9.8}{0.05}\displaystyle\int_{0.05}^{0.07} x\,dx$

$= 196 \left| \dfrac{x^2}{2} \right|_{0.05}^{0.07} = 98(0.07^2 - 0.05^2) = 0.2352$ Nm or J.

Practice Example - Cable

Ex: A 5-metre-long cable of mass 8 kg is used to lift a bucket off the ground. How much work is needed to raise the entire cable to height 5 m? Ignore the mass of the bucket and its contents.



$L = 5$ m

$M = 8$ kg.

Bucket is raised from $y=0$ to $y=5$

Mass density of cable $= \dfrac{M}{L} = \dfrac{8}{5} = 1.6$ kg/m.

When bucket is at a height 'y'

– Mass of Cable that remains to be lifted $= 1.6(5-y)$

→ ~~Mass of Cable subject to~~ Downward gravitational pull (Force) $= 1.6(5-y)\,g$

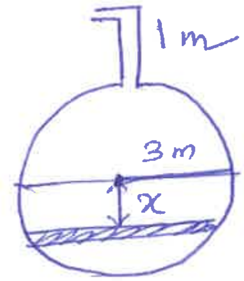
Work done $= \displaystyle\int_0^5 1.6(5-y)\,g\,dy = 1.6 \times 9.8 \left| 5y - \dfrac{y^2}{2} \right|_0^5$

$= 196$ J.

Practice Example - Water reservoir

Ex: A spherical tank of radius 3 metres is half-full of water. It has a spout of length 1 metre sticking up from the top of the tank. Find the work required to pump all of the water in the tank out the spout. The density of water is 1000 kilograms per cubic metre. The acceleration due to gravity is 9.8 metres per second squared.

$0 \le x \le 3$



x : Distance of disk from top surface of water.

Density, $\rho = 1000\,\text{kg}/\text{m}^3$

$g = 9.8\,\text{m}/\text{s}^2$

Disk: radius $= \sqrt{3^2 - x^2}$

area $= \pi(9 - x^2)$

Vol. $= \pi(9 - x^2)\,dx$

Mass $=$ Density $\times$ Vol.

$= 1000\pi(9 - x^2)\,dx$.

Practice Example - Water reservoir

Gravitational Force $=$ Mass $\times$ gravity

$= 9.8 \times 1000\pi(9 - x^2)\,dx$.

Work done on i^{th} disk $= 9800\pi(9 - x^2)\underbrace{(x+4)}_{\text{distance to top of spout}}\,dx$.

$\therefore$ Total work done $= \displaystyle\int_0^3 9800\pi(9 - x^2)(x+4)\,dx$

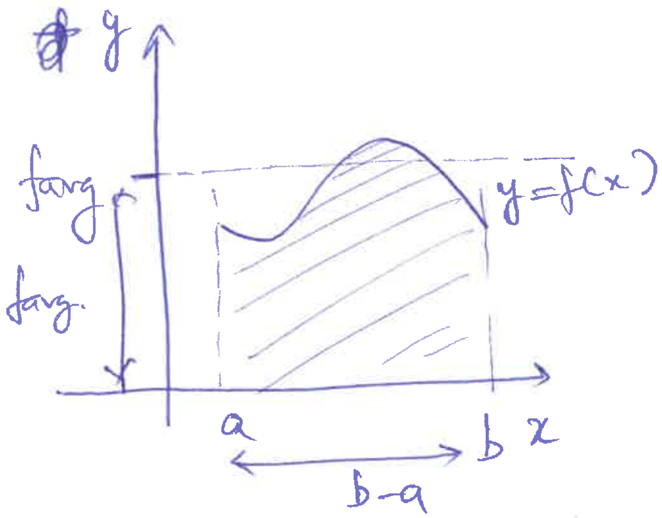
$= 9800\pi \times \dfrac{369}{4}$

$= 904050\pi$ Joules.

Average Value

- **Average Value of a Function** : For a continuous function f over $[a, b]$, the average value, y_{av}, is given by

$$f_{avg} = \boxed{y_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$



Area under $f(x)$ on $[a,b]$ $= \int_a^b f(x)\,dx$

$$f_{avg} \times (b-a) = \int_a^b f(x)\,dx.$$

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Examples

Ex: Find the average value over the given interval.

(a) $y = 1 + x^2$, $[-1, 3]$

$$y_{av} = \frac{1}{3-(-1)}\int_{-1}^{3}(1+x^2)\,dx = \frac{1}{4}\left|x + \frac{x^3}{3}\right|_{-1}^{3} = \frac{1}{4}\left[\left(3+\frac{3^3}{3}\right)-\left(-1-\frac{1}{3}\right)\right]$$

$$= \frac{1}{4}\left(12 + 4/3\right)$$

$$= 10/3 \,//$$

(b) $f(x) = 3\cos x$, $[-\pi/2, \pi/2]$

$$f_{avg} = \frac{1}{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)}\int_{-\pi/2}^{\pi/2} 3\cos x \, dx$$

$$= \frac{3}{\pi}\left|\sin x\right|_{-\pi/2}^{\pi/2} = \frac{3}{\pi}\left(\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{3}{\pi}(1+1) = 6/\pi$$

The MVT for Integrals

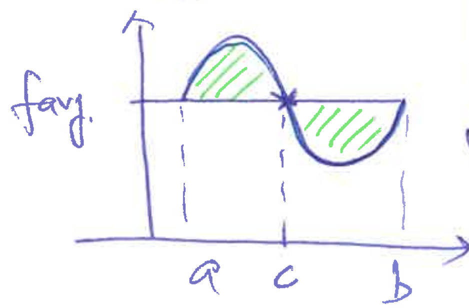
Theorem: If f is continuous on $[a, b]$, then there exists a number $'c'$ in $[a, b]$ such that

$$\boxed{f(c) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

that is,

MVT for Integrals: $$\boxed{\int_a^b f(x)\,dx = f(c)(b-a)}$$

$$\boxed{f(c) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$



MVT for Derivatives:

f(x) is cont, diff on (a,b)

There is a 'c' in (a,b) such that

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

"You can always chop off the top of (2D) mountains at a certain height and use it to fill the valleys so that the mountain becomes completely flat."

Example

Ex: If f is continuous and $\int_1^3 f(x)\,dx = 8$, show that f takes on the value 4 at least once on the interval $[1, 3]$.

$$\int_1^3 f(x)\,dx = 8$$

On $[1,3]$

$$f_{avg} = \frac{1}{3-1}\int_1^3 f(x)\,dx$$

$$= \frac{1}{\cancel{2}} \times \cancel{8}\,4$$

$$= 4 = f(c)$$

i.e, there is a 'c' in $[1,3]$ such that $f(c) = 4$ //

Example

Ex: Show that the average velocity of a car over a time interval $[t_1, t_2]$ is the same as the average of its velocities during the trip.

$s(t)$: displacement of car at time t ; Velocity $\boxed{v(t) = s'(t)}$

Avg. velocity over $[t_1, t_2]$ $= \dfrac{\text{Change in Position}}{\text{Change in time}} = \dfrac{s(t_2) - s(t_1)}{t_2 - t_1}$ — ①

From previous discussion, value of avg velocity $= V_{avg} = \dfrac{1}{t_2 - t_1} \displaystyle\int_{t_1}^{t_2} v(t)\,dt = \dfrac{1}{t_2 - t_1} \int_{t_1}^{t_2} s'(t)\,dt$

$= \dfrac{1}{t_2 - t_1} [s(t_2) - s(t_1)]$ — ②

Using Net Change thm.

$\therefore$ Using ① & ② $\boxed{\dfrac{s(t_2) - s(t_1)}{t_2 - t_1} = V_{avg}.}$

Practice Example

Ex: The linear density in a rod 8 m long is $\dfrac{12}{\sqrt{x+1}}$ kg/m where x is measured in meters from one end of the rod. Find the average density of the rod.

$[0, 8]$

Avg. density of the rod $= \dfrac{1}{8-0} \displaystyle\int_0^8 \frac{12}{\sqrt{x+1}}\,dx$

$= \dfrac{12}{8} \displaystyle\int_0^8 \frac{dx}{\sqrt{x+1}}$

$u = x+1$
$du = dx$
$x = 0 : u = 1$
$x = 8 : u = 9$

$= \dfrac{3}{2} \displaystyle\int_1^9 \frac{du}{\sqrt{u}}$

$= \dfrac{3}{\cancel{2}} \left| \dfrac{u^{1/2}}{1/\cancel{2}} \right|_1^9$

$= 3(\sqrt{9} - \sqrt{1}) = 3(3-1) = 6$ kg/m