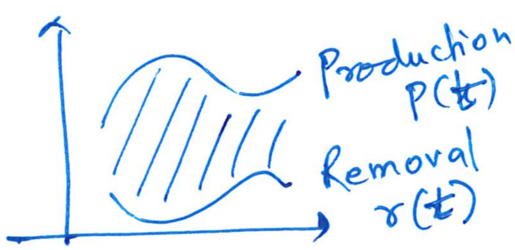
Chapter 6: Applications of Integration

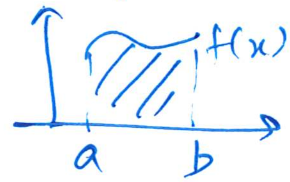
- Area between Curves
- Volumes
- Work
- Average Values



* FTC I & II

$$\int_{t_1}^{t_2} (\text{rate of change}) = \text{Net Change in the function}$$

* $\int_a^b f(x)\,dx = \text{Area under } f(x) \text{ on } [a, b]$



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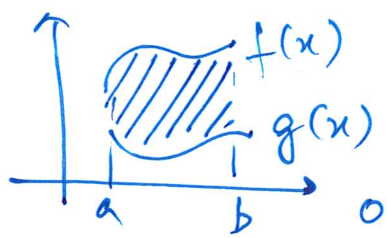

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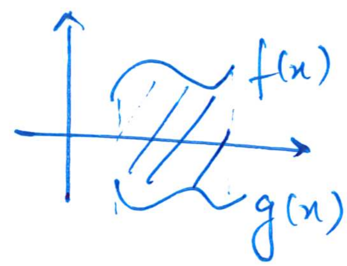
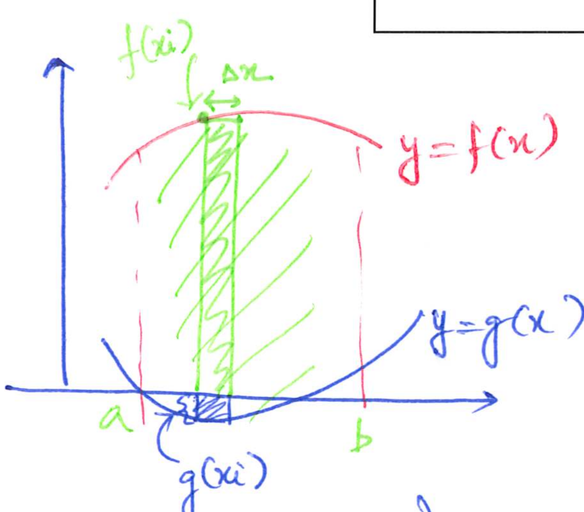
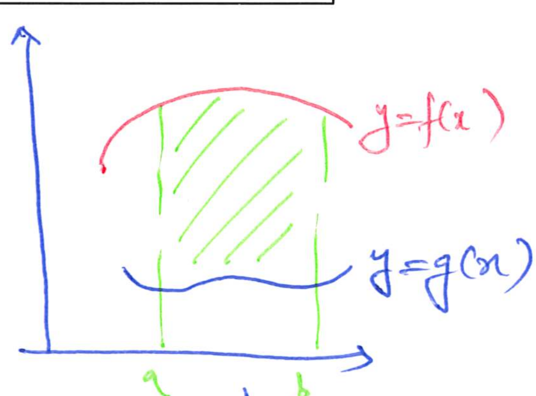


Area between Curves

- For any two continuous functions f, g, such that $f(x) \geq g(x)$ over $[a, b]$,

$$\boxed{\text{Area between the curves} = \int_a^b [f(x) - g(x)]\,dx}$$





$$\text{Area} = \int_a^b (f(x) - g(x))\,dx$$

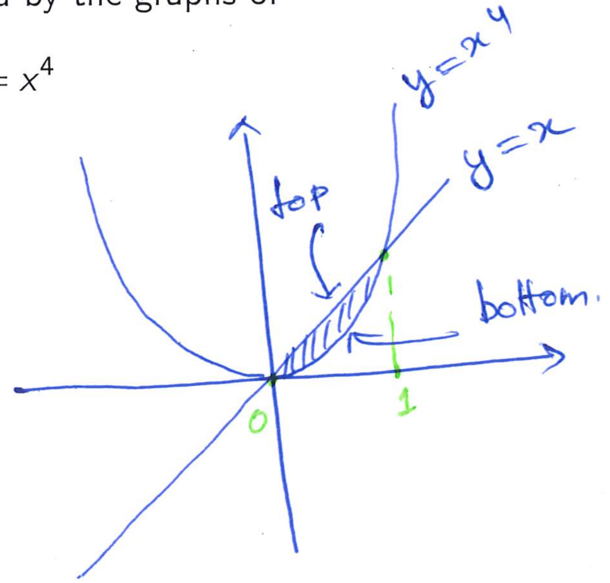
Height $= f(x_i) - g(x_i)$
Width $= \Delta x$

$$\to \text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\,\Delta x = \int$$

Example

$\boxed{\text{Ex:}}$ Find the area of the region enclosed by the graphs of

$$y = x, y = x^4$$



Area of the enclosed region

$$= \int_0^1 [x - x^4]\, dx$$

$$= \left| \frac{x^2}{2} - \frac{x^5}{5} \right|_0^1$$

$$= \left(\frac{1}{2} - \frac{1}{5}\right) - 0$$

$$= \frac{3}{10} \,//$$

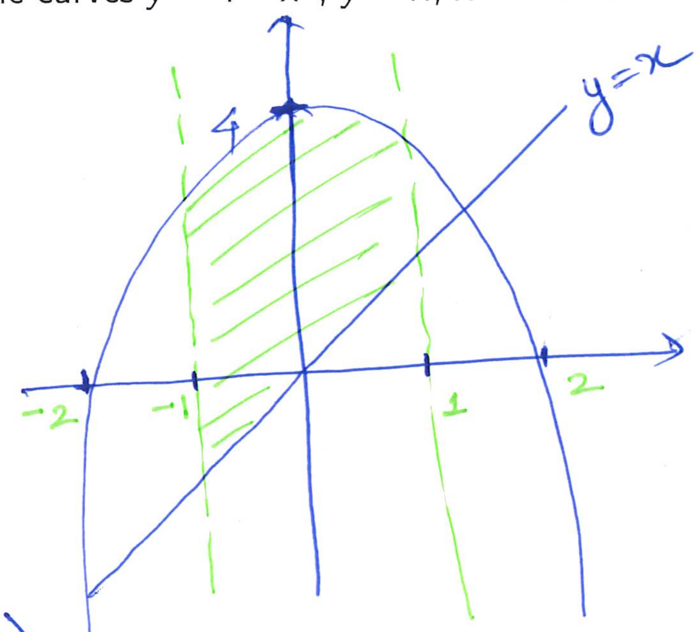
$$x^4 - x = 0$$

$$x(x^3 - 1) = 0$$

$\boxed{x = 0}$ $\quad x^3 - 1 = 0$ $\quad \boxed{x = 1}$

Example

$\boxed{\text{Ex:}}$ Find the area bounded by the curves $y = 4 - x^2$, $y = x$, $x = -1$ and $x = 1$.



Area enclosed

$$= \int_{-1}^{1} \Big[\underbrace{(4 - x^2)}_{\text{top curve}} - \underbrace{x}_{\text{bottom curve}} \Big]\, dx$$

$$= \left| 4x - \frac{x^3}{3} - \frac{x^2}{2} \right|_{-1}^{1}$$

$$= \left(4 - \frac{1}{3} - \cancel{\frac{1}{2}}\right) - \left(-4 + \frac{1}{3} - \cancel{\frac{1}{2}}\right)$$

$$= 2\left(4 - \frac{1}{3}\right) = \frac{22}{3} \,//$$

Example

Ex: Find the area of the region enclosed by the graphs of

Top curve $\rightarrow y = \sqrt{2x+6}$

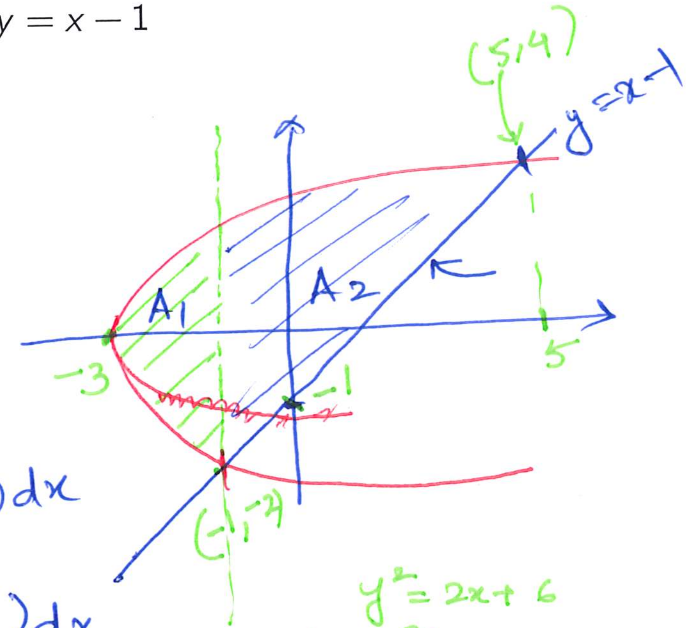
$$y^2 = 2x + 6, y = x - 1$$

$$\Downarrow$$

$$y = \pm\sqrt{2x+6}$$

Bottom Curve

$$B(x) = \begin{cases} -\sqrt{2x+6}, & -3 \le x \le -1 \\ x-1, & -1 \le x \le 5 \end{cases}$$



Area enclosed

$$= A_1 + A_2$$

$$= \int_{-3}^{-1} \left(\sqrt{2x+6} - \left(-\sqrt{2x+6}\right)\right) dx + \int_{-1}^{5} \left(\sqrt{2x+6} - (x-1)\right) dx$$

$$= 2\int_{-3}^{-1} \sqrt{2x+6}\, dx + \int_{-1}^{5} \sqrt{2x+6} - \int_{-1}^{5} (x-1)\, dx$$

$= 18$ (using substitution rule)

$y^2 = 2x + 6$

$y^2 - 6 = 2x$
$= 2(y+1)$

$y^2 - 2y - 8 = 0$

$(y+2)(y-4) = 0 \Rightarrow y = -2$, $y = 4$

Sideways Integration

In some cases, it may be simpler to divide the area horizontally, and construct an integral in 'y'.

Same example as above

Ex: Find the area of the region enclosed by the graphs of

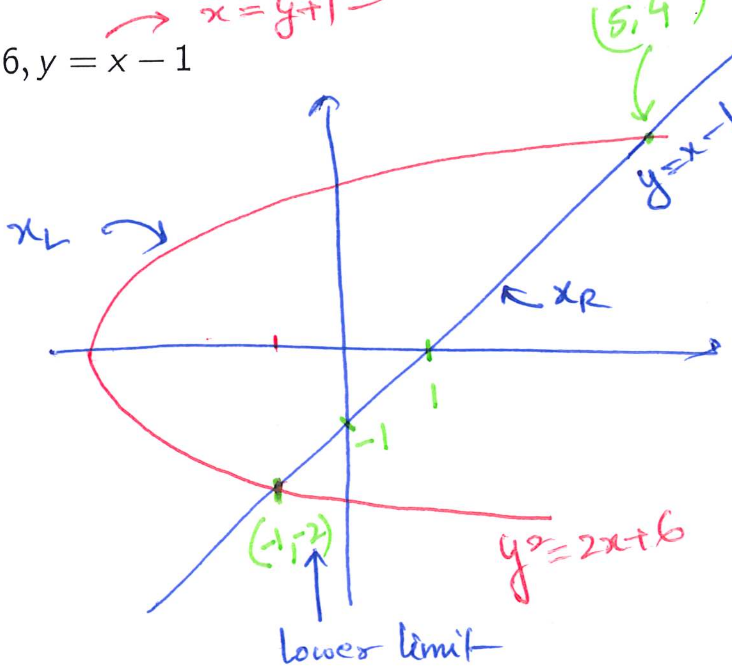
$$y^2 = 2x + 6, y = x - 1$$

$x = y + 1 \rightarrow \boxed{x_R = y + 1}$

$2x = y^2 - 6$

$x = \frac{y^2 - 6}{2}$

$$\boxed{x_L = \frac{1}{2}y^2 - 3}$$



$$\text{Area enclosed} = \int_{-2}^{4} (x_R - x_L)\, dy$$

$$= \int_{-2}^{4} \left(y + 1 - \frac{1}{2}y^2 + 3\right) dy$$

$$= 18$$

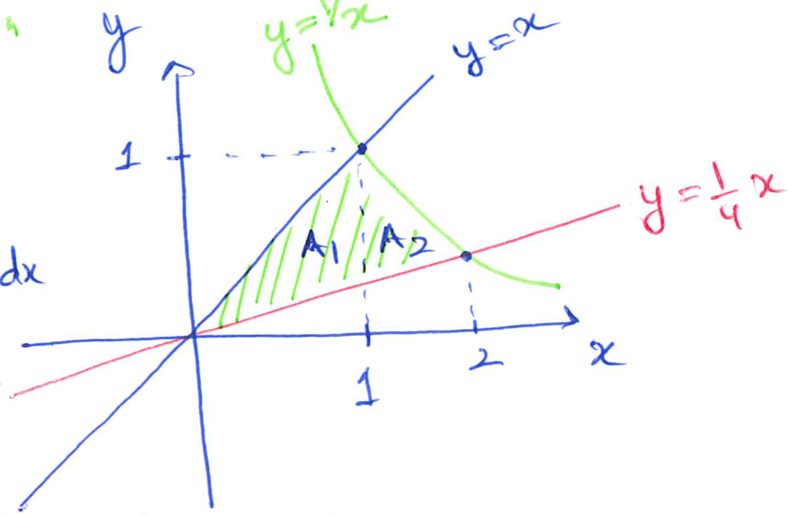
Example

$\boxed{\text{Ex:}}$ Find the area of the region enclosed by the curves $y = 1/x$, $y = x$ and $y = 1/4x$, using

(i) x as the variable of integration

(ii) y as the variable of integration

(i) x as the variable of integration



Area enclosed

$= A_1 + A_2$

$= \int_0^1 \left(x - \frac{1}{4}x\right) dx + \int_1^2 \left(\frac{1}{x} - \frac{1}{4}x\right) dx$

$= \int_0^1 \frac{3}{4} x \, dx + \left| \ln x - \frac{x^2}{8} \right|_1^2$

$= \left| \frac{3}{8} x^2 \right|_0^1 + \left(\ln 2 - \frac{2^2}{8}\right) - \left(\ln 1 - \frac{1}{8}\right) = \ln 2$

Example

(ii) y as the variable of integration

Area enclosed

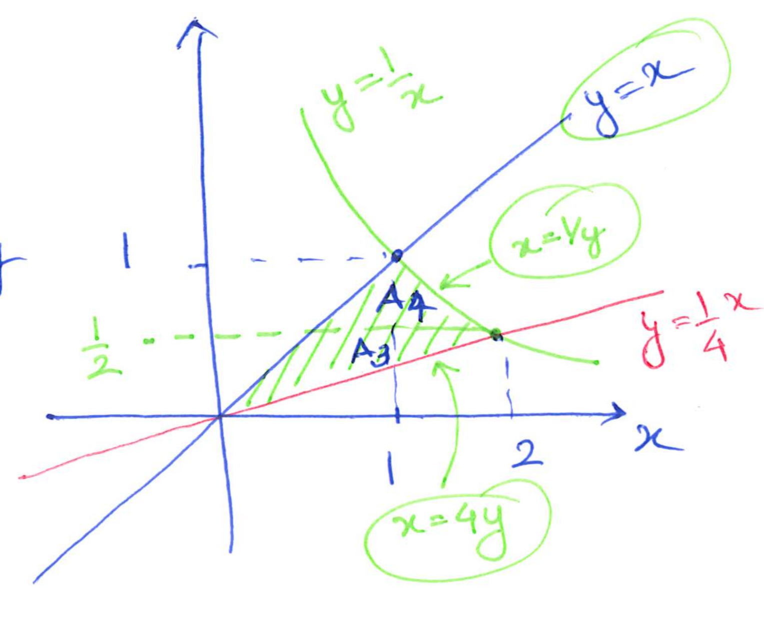
$= A_3 + A_4$

$= \int_0^{1/2} (4y - y)\, dy + \int_{1/2}^1 \left(\frac{1}{y} - y\right) dy$

$= \int_0^{1/2} 3y \, dy + \left| \ln y - \frac{y^2}{2} \right|_{1/2}^1$

$= \left| \frac{3y^2}{2} \right|_0^{1/2} + \left| \quad \right|$

$= \ln 2$



Practice Example

$\boxed{\text{Ex:}}$ Find the area bounded by $x = 2y^2$ and $x = 4 + y^2$.

Area enclosed

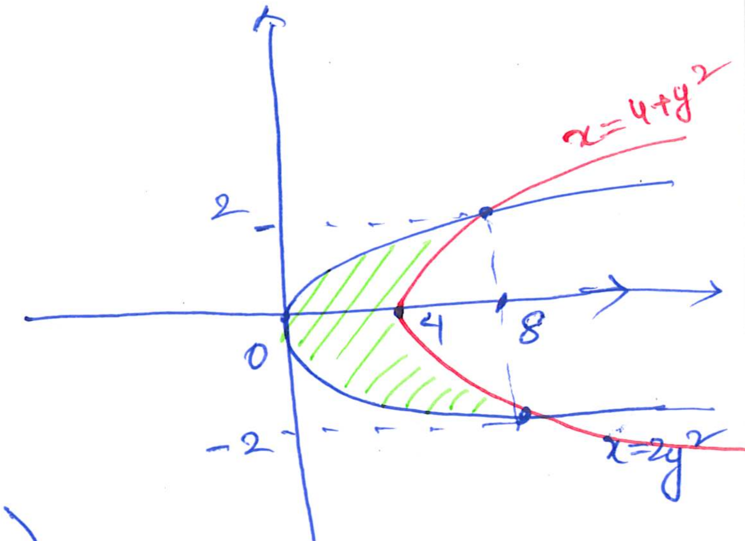
$$= \int_{-2}^{2} \left[(4+y^2) - (2y^2) \right] dy$$

$$= \int_{-2}^{2} (4 - y^2)\, dy$$

$$= \left| 4y - \frac{y^3}{3} \right|_{-2}^{2}$$

$$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)$$

$$= 2\left(\frac{16}{3}\right) = \frac{32}{3} \quad //$$



Practice Example

$\boxed{\text{Ex:}}$ Find the area of the region bounded by the graphs of

$$y = e^x, y = x^2 - 1$$

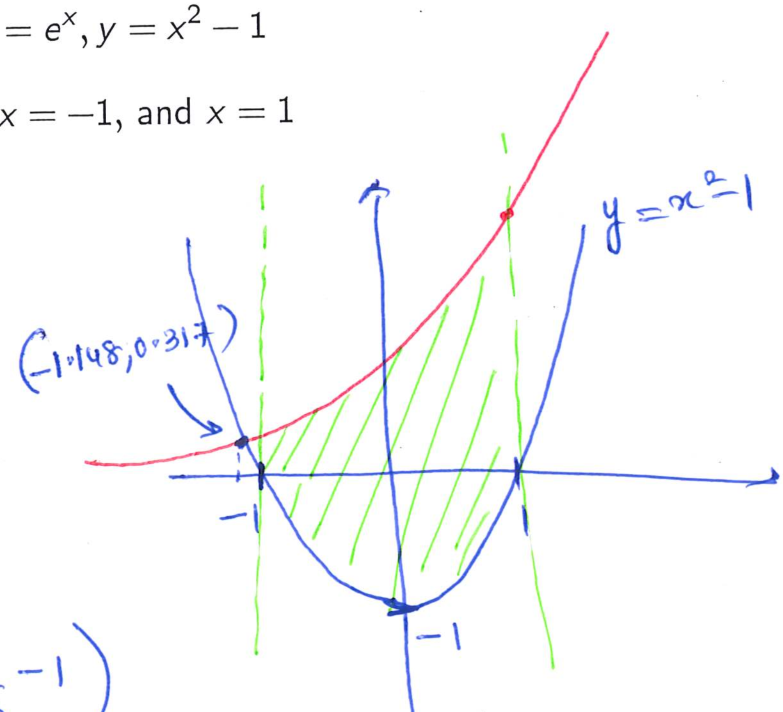
and bounded on the sides by $x = -1$, and $x = 1$

Area enclosed

$$= \int_{-1}^{1} \left[\underset{\substack{\uparrow \\ \text{top}}}{e^x} - \underset{\substack{\uparrow \\ \text{bottom.}}}{\underbrace{(x^2 - 1)}} \right] dx$$

$$= \left| e^x - \frac{x^3}{3} + x \right|_{-1}^{1}$$

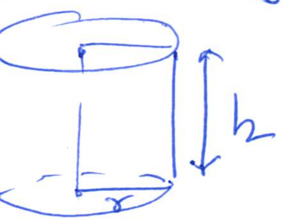
$$= \left(e^1 - \frac{1}{3} + 1\right) - \left(e^{-1} + \frac{1}{3} - 1\right) =$$



...lumes

...ple Shapes:

...ght Circular Cylinder

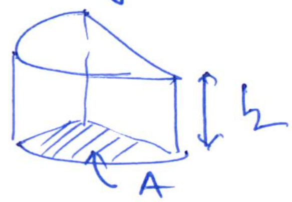


...lume = Area of base × Height

$= \pi r^2 \times h$

$= \pi r^2 h.$

Right Cylinder

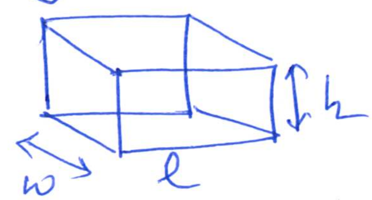


V = Area of base × height

$= Ah.$

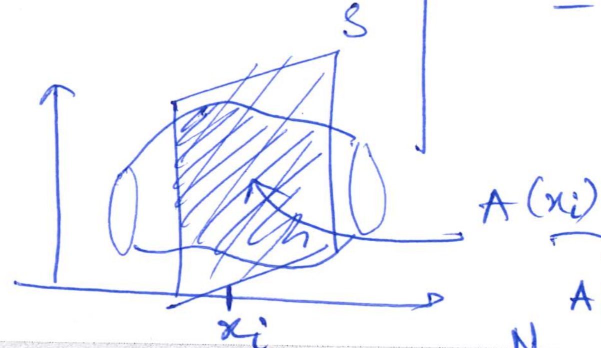
Cuboid

Rectangular box



$V = (l \times w) \times h$

$= lwh$



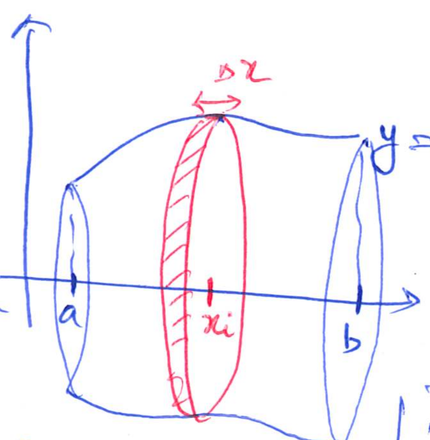
$A(x_i)$

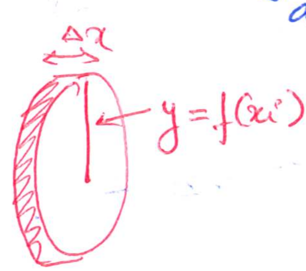
$A(x_i) . \Delta x$

$V \approx \sum_{i=1}^{N} A(x_i) \Delta x$

Volumes by Disk Method

$$\text{Volume} = \lim_{N \to \infty} \sum_{i=1}^{N} A(x_i) \Delta x = \int_a^b A(x)\,dx.$$





$\pi(\text{radius})^2 \cdot \text{thickness}$

Volume of the approximating disk/cylinder

$= A(x) . \Delta x$

$= \pi y^2 \Delta x$

$= \pi f(x)^2 \Delta x$

$y = f(x)$

$\downarrow$

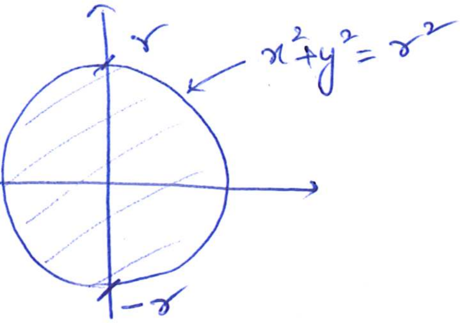
$x = g(y)$

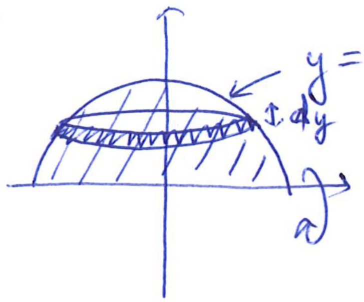
$$\therefore \text{Volume of Solid} = \int_a^b \pi f(x)^2 \Delta x$$

$$\text{Volume} = \int_a^b \pi x^2\,dy = \int_a^b \pi [g(y)]^2\,dy \;//$$

ample

Ex: Show that the volume of a sphere of radius r is $V = \dfrac{4}{3}\pi r^3$.



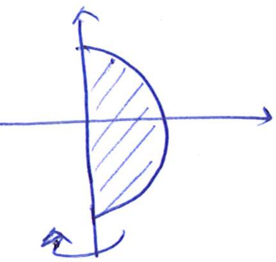


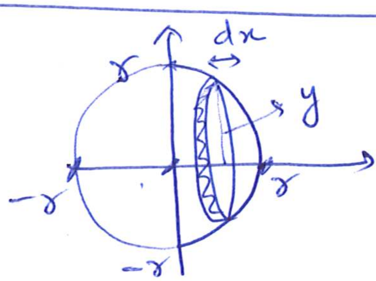
$y=\sqrt{r^2-x^2}=f(x) \Rightarrow$

Volume of disk

$= \pi x^2\, dy$

$= \pi (r^2-y^2)\, dy$





Volume of sphere

$= \int_{-r}^{r} \pi (r^2-y^2)\, dy$

$= 2\int_0^r (\pi r^2 - \pi y^2)\, dy$

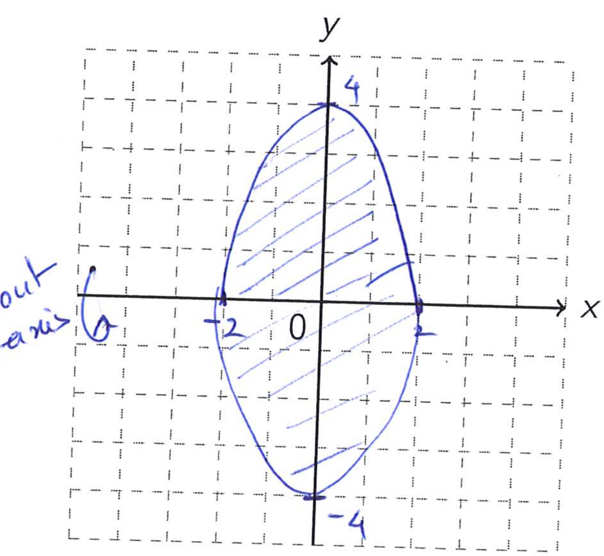
$= 2\cdot\left| \pi r^2 y - \pi \frac{y^3}{3} \right|_0^r$

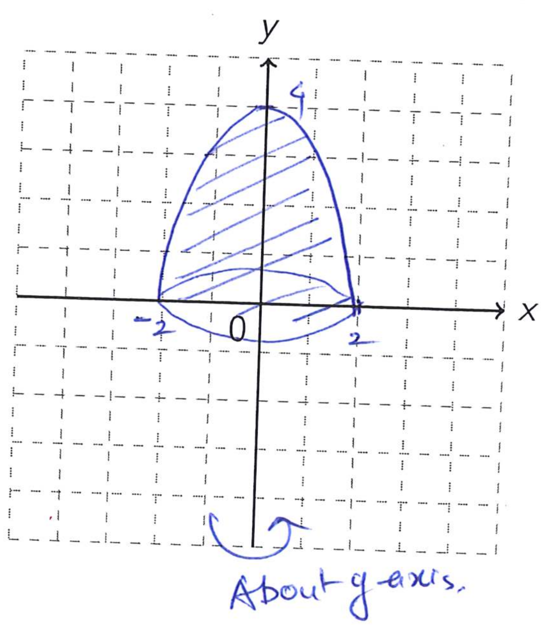
$= 2\pi\left(r^3-\frac{r^3}{3}\right) = 2\pi\cdot\frac{2}{3}r^3 = \frac{4}{3}\pi r^3$ //

Volumes of Revolution

The goal of this section is to use integrals to compute volumes of objects.

Ex: Sketch the graph of $y=-x^2+4$ and restrict the curve to above the x-axis. Revolve the curve about the x-axis and y-axis, and sketch the result.

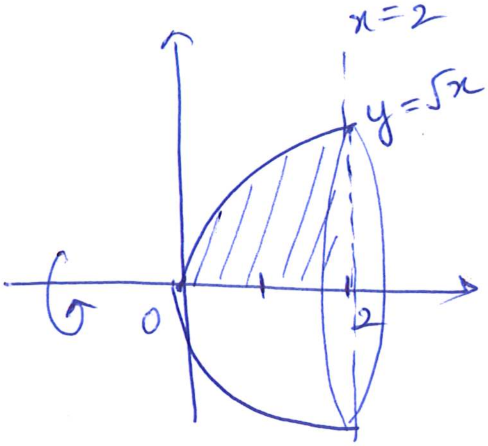


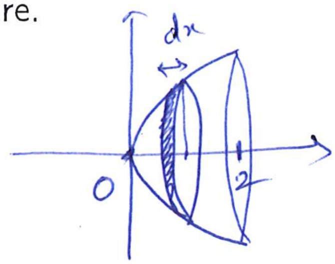


Example (about x-axis)

x axis

$\boxed{\text{Ex:}}$ An area is bounded by $y = 0, y = \sqrt{x}$, and $x = 2$. Find the volume generated by rotating the area around the $x-$axis. Illustrate the definition of volume by sketching a figure.





radius $= y$
$= f(x)$
$= \sqrt{x}$

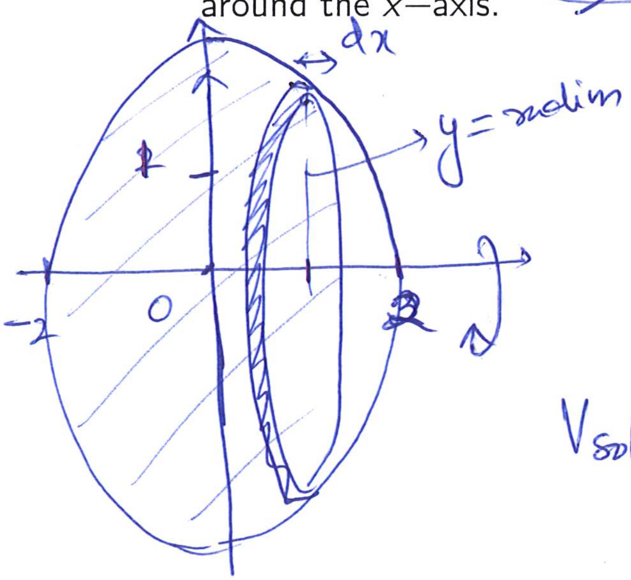
$V_{disk} = \pi\,(\text{radius})^2 \cdot \text{thickness}$
$= \pi\,(\sqrt{x})^2 \cdot dx$
$= \pi\, x\, dx$

$$V_{solid} = \int_0^2 \pi x\, dx$$
$$= \pi \int_0^2 x\, dx$$
$$= \pi \left| \frac{x^2}{2} \right|_0^2 = \frac{\pi}{2}\left(2^2 - 0^2\right) = \frac{\pi}{2} \cdot 4$$
$$= 2\pi$$

Example (about x-axis)

$\boxed{\text{Ex:}}$ Find the volume of revolution produced by rotating the region bounded by the curve, $\cancel{y = x^2 - 2x + 2}$, and the $x-$axis between $\cancel{[0, 3]}$, around the $x-$axis.

$y = -x^2 + 4$ $[0, 2]$

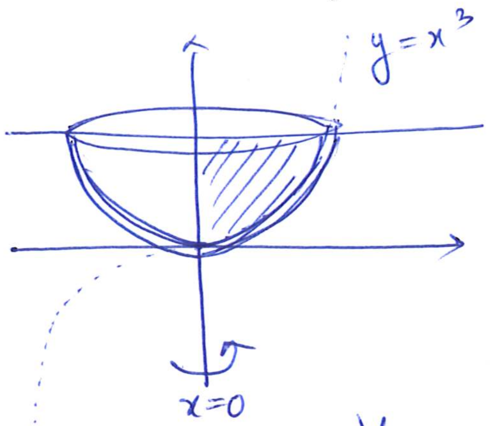


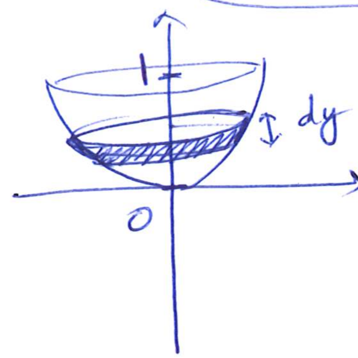
$V_{disk} = \pi\,(\text{radius})^2 (\text{thickness})$
$= \pi y^2 dx$
$= \pi (-x^2 + 4)^2 dx$

$$V_{solid} = \int_{-2}^{2} \pi (-x^2+4)^2 dx$$
$$= 2\pi \int_0^2 (+x^4 + 16 - 8x^2)\, dx$$
$$= 2\pi \left| \frac{x^5}{5} + 16x - \frac{8}{3}x^3 \right|_0^2 =$$

Example (about y-axis)

$\boxed{\text{Ex:}}$ Find the volume of revolution produced by rotating the region bounded by the curve, $y = x^3$, $y=1$ and the $x = 0$ around the y—axis.





$y = x^3$

$x = y^{1/3}$

$x^2 = y^{2/3}$

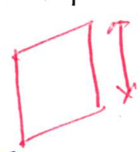
$$V_{disk} = \pi\,(\text{radius})^2\,dy = \pi\, x^2\, dy = \pi\, y^{2/3}\, dy$$

$$V_{solid} = \int_0^1 \pi y^{2/3}\,dy = \pi \left| \frac{y^{2/3+1}}{2/3+1} \right|_0^1 = \frac{3\pi}{5} \left| y^{5/3} \right|_0^1 = \frac{3\pi}{5} \;//$$

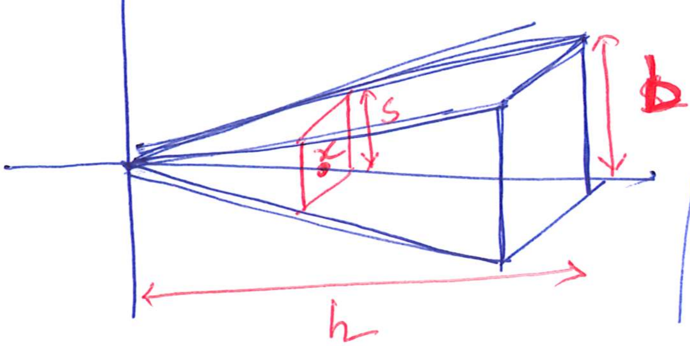
Volumes by using a cross-sectional area

Another way to use integrals for volume calculations is in cases where one can describe the cross-section perpendicular to the direction of integration.

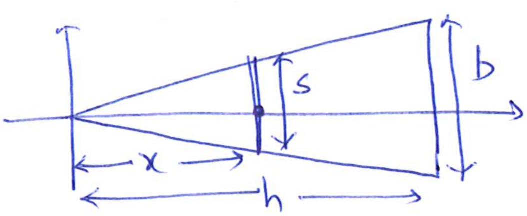
$\boxed{\text{Ex:}}$ Find the volume of a pyramid with square base with side length b and height h.



$s \longrightarrow \text{Area} = s^2 = \frac{x^2 b^2}{h^2} = A(x)$



$$\text{Volume} = \int_0^h A(x)\,dx = \frac{b^2}{h^2}\int_0^h x^2\,dx = \frac{b^2}{h^2}\left|\frac{x^3}{3}\right|_0^h = \frac{b^2}{h^2}\cdot\frac{1}{3}h^3 = \frac{1}{3}b^2 h \;//$$



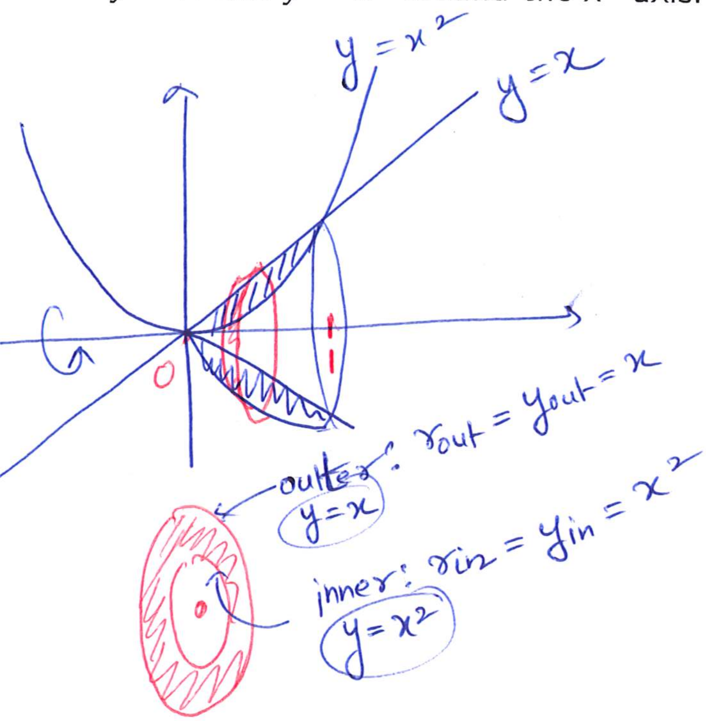
Using similar triangles,

$$\frac{x}{h} = \frac{s}{b}$$

$$\boxed{s = \frac{xb}{h}}$$

Example - Hollow Core

Ex: Find the volume of solid obtained by rotating the area bounded by $y = x$ and $y = x^2$ around the x-axis.



Area of washer (hollow disc) $= \pi (r_{out})^2 - \pi (r_{in})^2$

$$A(x) = \pi(x^2) - \pi(x^4)$$

Volume of washer $= A(x)\,dx$

Volume of solid

$$= \int_0^1 A(x)\,dx$$

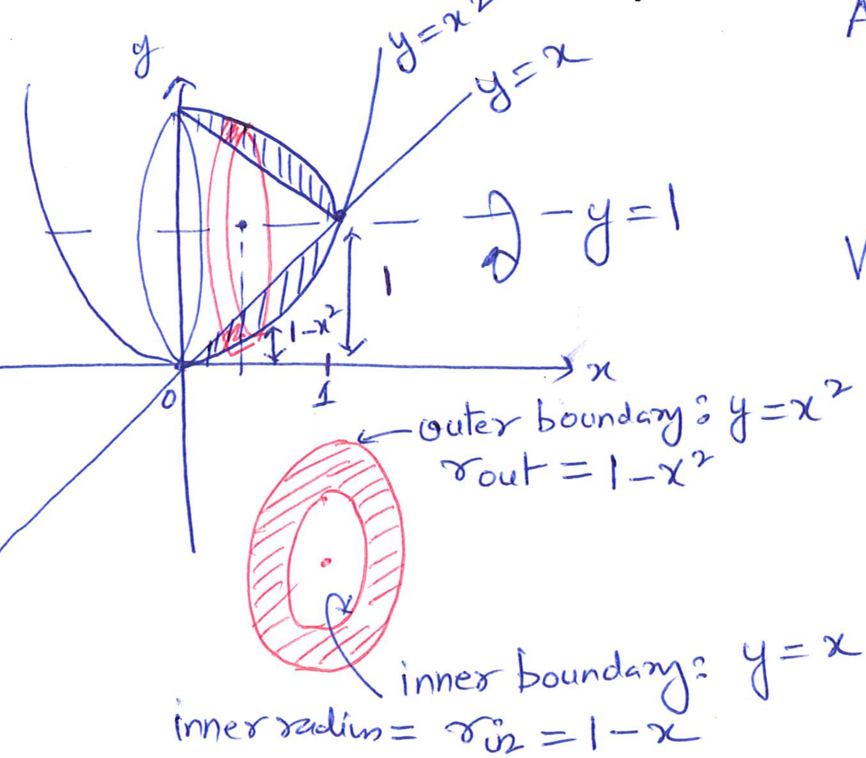
$$= \int_0^1 \pi(x^2 - x^4)\,dx$$

$$= \pi \left| \frac{x^3}{3} - \frac{x^5}{5} \right|_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$= \frac{2}{15}\pi \ //$$

Example - Hollow Core

Ex: Find the volume of solid obtained by rotating the area bounded by $y = x$ and $y = x^2$ around the line $y = 1$.



Area of washer $= \pi(r_{out})^2 - \pi(r_{in})$

$$A(x) = \pi(1-x^2)^2 - \pi(1-x)^2$$

Volume of solid

$$= \int_0^1 A(x)\,dx$$

$$= \pi \int_0^1 [(1-x^2)^2 - (1-x)^2]\,dx$$