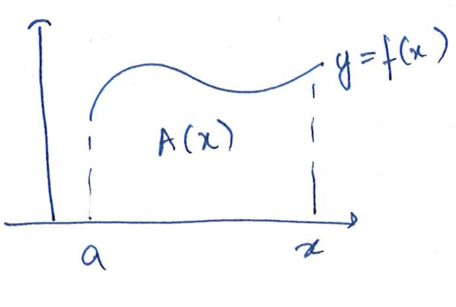
Chapter 5: Integrals

- Introduction to Integrals
- The Definite Integral and its Properties
- The Fundamental Theorem of Calculus
- The Substitution Rule

Area as a Function

Consider a function representing area under the graph of the curve $y = f(x)$ from a fixed point $'a'$ to a varying point $'x'$. What happens to the area as $'x'$ increases?

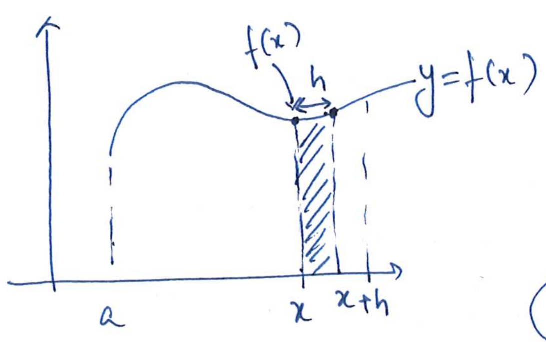


Area Function $\left[A(x) = \int_a^x f(x)\,dx \right.$; $A(x+h) = \int_a^{x+h} f(x)\,dx$

$$A(x+h) - A(x) \simeq f(x)\cdot h$$

$$\frac{A(x+h) - A(x)}{h} \simeq f(x)$$

$$\lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h\to 0} f(x)$$



$$\boxed{A'(x) = f(x)}$$

i.e, $A(x)$ is the anti-derivative of $f(x)$.

Fundamental Theorem of Calculus (Part I)

Let $a < b$ and let $f(x)$ be a function defined and continuous on $[a, b]$.

Theorem:Let

$$F(x) = \int_a^x f(t)\,dt$$

for any x in $[a, b]$. Then the function $F(x)$ is differentiable and further

$$F'(x) = f(x)$$

Ex: Find the derivative of the function $g(x)$, where

$$g(x) = \int_a^x \sqrt{1+t^2}\,dt$$

$g'(x) = \sqrt{1+x^2}$

Anti-differentiation

"Anti-differentiation is the reverse of differentiation."

For example, if we consider the function $f(x) = 2x$, then what is its anti-derivative?

$\frac{d}{dx}(\ ?\ ) = 2x$ → x^2, x^2+5, x^2-100 → x^2+C Most general anti-derivative.

Definition: If F is an antiderivative of f on an interval **I**, then the most general antiderivative of f on I is $F(x) + C$ or in other words,

$$\frac{d}{dx}[F(x) + C] = f(x)$$

where C is an arbitrary constant. This relationship can also be expressed as

$$\int f(x)\,dx = F(x) + C$$

Indefinite Integral notation

Most general anti-derivative

$F(x)$: Particular anti-derivative

Anti-Differentiation Rules

Rule	Function	Anti-derivative
Constant Rule	k	kx
Constant Factor Rule	$kf(x)$	$kF(x)$
Sum Rule	$f(x)+g(x)$	$F(x)+G(x)$
Power Rule	x^n	$\frac{x^{n+1}}{n+1}$
Natural Log Func.	$\frac{1}{x}$	$\ln\lvert x\rvert$
Natural Exp. Func.	e^{ax}	e^{ax}/a
Exp. Func.	b^x	$b^x/\ln b$
Cosine Rule	$\cos(ax)$	$\sin(ax)/a$
Sine Rule	$\sin(ax)$	$-\cos(ax)/a$
Tan Rule	$\sec^2(ax)$	$\tan(ax)/a$
Arctan Rule	$\frac{1}{1+x^2}$	$\arctan(x)$

Anti-derivatives - Calc I Review

Ex: Fill in the blank boxes:

f	f'		f	f'
$2x$	2		$\cos(x)$	$-\sin x$
x^3	$3x^2$		$2\sin 5x$	$\frac{2}{5}\cos(5x)$
$\frac{x^5}{5}+C$	x^4		e^x	e^x
$2-x^5$	$-5x^4$		e^{-3x}	$-3e^{-3x}$
$\frac{-6x^2}{2}$	$-6x$		$\frac{x^{4/3}}{4/3}+C$	$x^{1/3}$
$\ln x$	$1/x$		$2\cdot\frac{x^{2/3}}{2/3}+C$	$2x^{-1/3}$

Examples (FTC I)

Ex: Evaluate $\frac{d}{du}\left(\int_0^u \sin(1+t^3)\,dt\right)$

$$= \sin(1+u^3)$$

$$A(x) = \int_a^x f(t)\,dt$$
$$\downarrow$$
$$A'(x) = f(x)$$

Ex: Find the derivative of the function $f(x)$, where

$$f(x) = \int_1^{x^4} \sec t\,dt$$

$u = x^4$

$\frac{du}{dx} = 4x^3$

$$f'(x) = \frac{d}{dx}\left(\int_1^{x^4} \sec t\,dt\right)$$
$$= \frac{d}{dx}\left(\int_1^{u} \sec t\,dt\right)$$
$$= \frac{d}{du}\left(\int_1^{u} \sec t\,dt\right)\cdot\frac{du}{dx} \quad \leftarrow \text{Chain Rule}$$
$$= \sec(u)\cdot 4x^3$$
$$= \sec(x^4)\cdot 4x^3 \;//$$

Indefinite Integral

The **indefinite integral** of $f(x)$ denoted by $\int f(x)\,dx$ is the most general anti-derivative of $f(x)$.

Ex: Evaluate the following indefinite integrals.

(a) $\int (x^2+1)\,dx = \frac{x^3}{3} + x + C \;//$

(b) $\int (\sqrt{x}-1)\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - x + C = \frac{2}{3}x^{3/2} - x + C \;//$ ($\sqrt{x} \to x^{1/2}$)

(c) $\int e^{2t}\,dt = \frac{e^{2t}}{2} + C$

Table of Indefinite Integrals

(Similar to anti-der. rules table, just different notation)

$\int cf(x)\,dx = c\int f(x)\,dx$	$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
$\int k\,dx = kx + C$	
$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\int e^x\,dx = e^x + C$	$\int b^x\,dx = \frac{b^x}{\ln b} + C$
$\int \sin x\,dx = -\cos x + C$	$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$
$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$
~~$\int \sinh x\,dx = \cosh x + C$~~	~~$\int \cosh x\,dx = \sinh x + C$~~

Fundamental Theorem of Calculus (Part II)

Theorem: Let $G(x)$ be any function defined and continuous on $[a, b]$. Further $G(x)$ is differentiable such that $G'(x) = f(x)$ for all $a < x < b$. Then

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

$G(x)$ is the anti-derivative of $f(x)$

or equivalently,

$$\int_a^b G'(x)\,dx = G(b) - G(a)$$

FTC I:

$$\int_a^x f(x)\,dx = G(x) + C$$

$$\int_a^a f(x)\,dx = G(a) + C$$

$$0 = G(a) + C$$

$$C = -G(a)$$

Notation: $\int_a^b f(x)\,dx = \left| G(x) \right|_a^b = G(b) - G(a)$

FTC II: $\int_a^b f(x)dx = G(b) - G(a) \longleftarrow \int_a^x f(x)dx = G(x) - G(a)$

Thus, the two parts of FTC say that integration and differentiation are inverse processes.

Examples

Ex: Evaluate the following definite integrals:

(a) $\int_0^1 x^2\,dx$

$= \left| \frac{x^3}{3} \right|_0^1$

$= \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$ //

(b) $\int_0^2 e^{-3x}\,dx$

$= \left| \frac{e^{-3x}}{-3} \right|_0^2$

$= \frac{e^{-6}}{-3} - \left(\frac{e^0}{-3}\right)$

$= -\frac{1}{3}e^{-6} + \frac{1}{3}(1) = \frac{1}{3} - \frac{1}{3}e^{-6}$ //

(c) $\int_0^{\pi/4} (\sec\theta\tan\theta)\,d\theta$

$= \left| \sec\theta \right|_0^{\pi/4}$

$= \sec\left(\frac{\pi}{4}\right) - \sec(0)$

$= \sqrt{2} - 1$ //

(d) $\int_0^{\pi} (5e^x + 3\sin x)\,dx$

$= \left| 5e^x + 3(-\cos x) \right|_0^{\pi}$

$= \left| 5e^x - 3\cos x \right|_0^{\pi}$

$= (5e^{\pi} - 3\cos\pi) - (5e^0 - 3\cos 0)$

$= 5e^{\pi} - 3(-1) - 5(1) + 3(1)$

$= 5e^{\pi} + 3 - 5 + 3$

$= 5e^{\pi} + 1$ //

Practice Examples

Ex: Evaluate the following integrals:

(a) $\int_0^1 \frac{4}{1+p^2}\,dp$

$= 4\left| \tan^{-1} p \right|_0^1$

$= 4(\tan^{-1} 1 - \tan^{-1} 0)$

$= 4\left(\frac{\pi}{4} - 0\right) = \pi$ //

(b) $\int_0^1 x(\sqrt{x} + \sqrt[4]{x})\,dx$

$= \int_0^1 (x^{3/2} + x^{1+1/4})\,dx$

$= \int_0^1 (x^{3/2} + x^{5/4})\,dx = \left| \frac{x^{5/2}}{5/2} + \frac{x^{9/4}}{9/4} \right|_0^1$

$= \frac{1}{5/2} + \frac{1}{9/4} = \frac{2}{5} + \frac{4}{9}$ //

(c) $\int_0^{\pi/4} \frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta$

$= \int_0^{\pi/4} \left(\frac{1}{\cos^2\theta} + 1\right)d\theta$

$= \int_0^{\pi/4} (\sec^2\theta + 1)\,d\theta = \left| \tan\theta + \theta \right|_0^{\pi/4}$

$= \left(\tan\frac{\pi}{4} + \frac{\pi}{4}\right) - (\tan 0 + 0)$

$= 1 + \frac{\pi}{4} - 0 = 1 + \pi/4$ //

(d) $\int_0^{\sqrt{3}/2} \frac{dr}{\sqrt{1-r^2}}$

$= \left| \sin^{-1} r \right|_0^{\sqrt{3}/2}$

$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}(0)$

$= \frac{\pi}{3} - 0 = \pi/3$ //

Applications of FTC

Ex: Evaluate the area under the graph of the following functions over the given interval:

(a) $f(x) = \frac{1}{x^2}$, $[1, 10]$

$$\text{Area} = \int_1^{10} \frac{1}{x^2}\,dx$$
$$= \int_1^{10} x^{-2}\,dx$$
$$= \left| \frac{x^{-2+1}}{-2+1} \right|_1^{10}$$
$$= \left| -x^{-1} \right|_1^{10}$$
$$= \left| -\frac{1}{x} \right|_1^{10} = -\frac{1}{10} - \left(-\frac{1}{1}\right)$$
$$= -\frac{1}{10} + 1 = \frac{9}{10}$$

(b) $g(x) = x^{-2/3}$, $[1, 8]$

$$\text{Area} = \int_1^{8} x^{-2/3}\,dx$$
$$= \left| \frac{x^{-2/3+1}}{-2/3+1} \right|_1^{8}$$
$$= \left| 3\, x^{1/3} \right|_1^{8}$$
$$= 3 \cdot 8^{1/3} - 3 \cdot 1^{1/3}$$
$$= 3(2) - 3$$
$$= 6 - 3$$
$$= 3$$

Applications of FTC

Ex: Evaluate the derivatives of the following functions

(a) $\int_0^{x^2} e^{-t^2}\,dt$

$u = x^2$

$\frac{du}{dx} = 2x$

$$\frac{d}{dx}\left(\int_0^{x^2} e^{-t^2}\,dt \right) = ?$$
$$= \frac{d}{dx}\left(\int_0^{u} e^{-t^2}\,dt \right)$$
$$= \frac{d}{du}\left(\int_0^{u} e^{-t^2}\,dt \right) \cdot \frac{du}{dx}$$
$$= e^{-u^2} \cdot \frac{du}{dx}$$
$$= e^{-x^4} \cdot 2x$$
$$= 2x e^{-x^4}$$

(b) $\int_1^{e^x} \ln t\,dt$

$x = \ln u$

$u = e^x$

$\frac{du}{dx} = e^x$

$$\frac{d}{dx}\left(\int_1^{e^x} \ln t\,dt \right)$$
$$= \frac{d}{dx}\left(\int_1^{u} \ln t\,dt \right)$$
$$= \frac{d}{du}\left(\int_1^{u} \ln t\,dt \right) \cdot \frac{du}{dx}$$
$$= \ln u \cdot \frac{du}{dx}$$
$$= x \cdot (e^x) = x e^x$$

Net Change Theorem

The integral of a rate of change ($f(x)$) is the net change:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Applications:

- $\int_{t_1}^{t_2} (\text{Velocity})\,dt = \Delta\text{Position}$ = Change in Position = Displacement
- $\int_{t_1}^{t_2} (\text{Acceleration})\,dt = \Delta\text{Velocity}$ = [Net] Change in Velocity over $[t_1, t_2]$
- $\int_{t_1}^{t_2} (\text{Flow Rate})\,dt = \Delta\text{Volume}$ = Net Change in Volume over $[t_1, t_2]$
- $\int_{t_1}^{t_2} (\text{Rate of Population Growth})\,dt = \Delta\text{Population}$ etc...

= Net change in population over $[t_1, t_2]$
(total)

Application

Ex: If $w'(t)$ is the rate of growth of a child in pounds per year, what does $\int_5^{10} w'(t)\,dt$ represent?

This integral represents the total (net) growth of the child's [weigh] in between 5 to 10 years of age.
OR, The increase in the child's weight (in pounds) between the ages of 5 and 10.

Ex: If $f(x)$ is theslope of a trail at a distance of xmiles from the start of the trail, what does $\int_3^5 f(x)\,dx$ represent?

This integral represents the ~~net change in~~ change in elevation of the trail between 3 and 5 miles from the start.

Example

Ex: Position of a car along the highway is given by, $x(t) = 100t^2 + 5t$, where $x(t)$ is in km, t is in hours. Find the total displacement of the car in one hour.

Displacement in 1 hour $= x(1) - x(0)$
$= 100 \cdot 1^2 + 5 \cdot 1 - 0$
$= 105\,km$ //

Ex: Speed of a car along the highway is given by, $v(t) = 100t^2 + 5t$, in km/h, where $0 < t < 1$. Find the total displacement of the car in one hour.

Total displacement in 1 hour

$$= \int_0^1 v(t)\,dt$$

$$= \int_0^1 (100t^2 + 5t)\,dt$$

$$= \left| 100 \cdot \frac{t^3}{3} + 5\frac{t^2}{2} \right|_0^1 = \frac{100}{3}(1^3) + \frac{5}{2}(1^2) - 0 = 35.83\,km$$ //

Example

Ex: Two cars labelled 1 and 2 start side by side and accelerate from rest. The graph shows their velocity functions, t is in minutes.

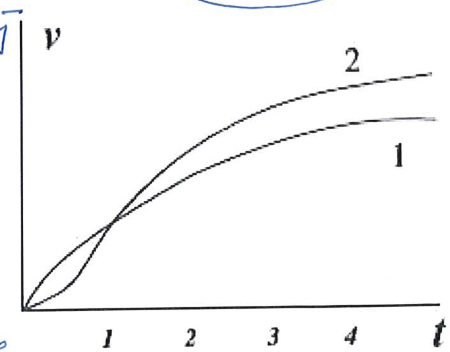
(a) At what time(s) do the cars have equal velocities? $t=0, t=1$

(b) At what time(s) do the cars have equal accelerations?

(c) Which car is ahead after 1 minute? Which car is ahead after 3 minutes?

(d) When does one car overtake the other? → around $t \sim 2$ mins — when the areas under both curves are equal.

(b) Equal acceleration ⇒ Equal rate of change of velocity
⇒ Equal slopes of tangents to the two curves
⇒ $t \sim 3$



(c) After 1 min → Car 1 is ahead; After 3 mins → Car 2 is ahead } Displacement is area under the graph. //

Application

Ex: An express mail truck delivers mail to various companies situated along a central avenue and often goes back and forth as new mail arrives. Over some period of time, $0 < t < 10$, its velocity (in km per hour) is given by $v(t) = t^2 - 9t + 14 = (t-2)(t-7)$

(a) Find the displacement over this period of time.

(b) How much gasoline was consumed during this period of time if the vehicle uses 1/10 litres per km.

(a) Displacement $= \int_0^{10} v(t)\,dt$

$= \int_0^{10} (t^2 - 9t + 14)\,dt$

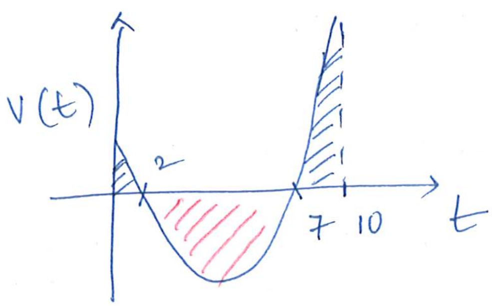
$= \left| \frac{t^3}{3} - \frac{9t^2}{2} + 14t \right|_0^{10}$

$= \frac{1}{3}(10)^3 - \frac{9}{2}(10)^2 + 14(10) - 0$

$\simeq 23.33$ km

Application

(b) Total distance travelled $= \int_0^2 v(t)\,dt - \int_2^7 v(t)\,dt + \int_7^{10} v(t)\,dt$



↑ artificially introduced negative sign to nullify the change in direction of the truck's motion

$= \left| \frac{t^3}{3} - 4.5t^2 + 14t \right|_0^2 - \left| \frac{t^3}{3} - 4.5t^2 + 14t \right|_2^7 + \left| \quad \right|_7^{10}$

$= 12.67 - (-8.16) + 227.5$

$\simeq 248.33$ km.

$\therefore$ Amount of gasoline consumed $= \frac{1}{10} \times 248.33$

$= 24.8$ litres

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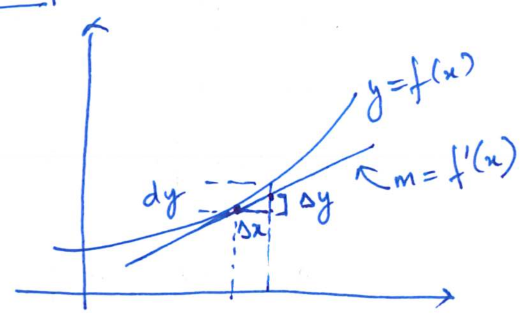
Differentials- Calculus I

From *Differential Calculus,* we know that

$$\text{Slope of line, } m = \frac{\text{chage in y}}{\text{change in x}} = \frac{\Delta y}{\Delta x}$$

which is same as writing, $\Delta y = m \cdot \Delta x$



For sufficiently small step-sizes ($\Delta x \to 0$),

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\boxed{dy = f'(x)dx}$$

where $f'(x)$ is the slope of tangent line (derivative), and the quantities dy and dx are called <u>differentials</u>.

Rules for derivatives and differentials

Ex: Calculate the differentials of the following functions

(i) $y = x^5$

$dy = 5x^4 dx$

(ii) $y = \sin x$

$dy = \cos x \, dx$

(iii) $y = e^{2x}$

$dy = 2e^{2x} dx$

(iv) $y = \ln x$

$dy = \frac{1}{x} dx$

(v) $y = x^2 + 5x$

$dy = (2x+5) dx$

(vi) $y = e^{x^2}$

$dy = e^{x^2} . 2x \, dx$

Anti-differentiation and indefinite integrals

We know $\int x^2 \, dx = \frac{x^3}{3} + C$

Instead, if we now want to evaluate

$\int (x+1)^2 \, dx = \int u^2 du = \frac{u^3}{3} + c = \frac{(x+1)^3}{3} + c$

$u = x+1$, $du = dx$

We know $\int \cos x \, dx = \sin x + C$

Instead, evaluate $\int \cos(x+2) \, dx = \int \cos u \, du = \sin u + c$
$= \sin(x+2) + c$

$u = x+2$
$du = dx$

We know $\int \sqrt{x} \, dx = \frac{x^{3/2}}{3/2} + C = \frac{2}{3} x^{3/2} + C$ //

Instead, evaluate $\int \sqrt{x-5} \, dx = \int \sqrt{u} \, du = \frac{2}{3} u^{3/2} + c = \frac{2}{3} (x-5)^{3/2} + c$ //

$u = x-5$
$du = dx$

Simple Substitution

Alt. way $\rightarrow \int (5x+1)^2 dx = \int (25x^2+1+10x)\,dx = ?$

(i) $\int (5x+1)^2\, dx =$ $\int u^2 \cdot \frac{du}{5} = \frac{1}{5}\int u^2 du = \frac{u^3}{5 \cdot 3} + C = \frac{u^3}{15} + C = \frac{(5x+1)^3}{15} + C$

$u = 5x+1$
$du = 5dx$
$\frac{du}{5} = dx$

(ii) $\int 2x\sqrt{x^2+1}\, dx =$ $\int \underbrace{\sqrt{x^2+1}}_{\sqrt{u}} \cdot \underbrace{2x\,dx}_{du} = \int \sqrt{u}\, du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(x^2+1)^{3/2} + C$

$u = x^2+1$
$du = 2x\,dx$

(iii) $\int \underbrace{\sin^2 x}_{u^2} \underbrace{\cos x\, dx}_{du} =$ $\int u^2 du = \frac{u^3}{3} + C = \frac{1}{3}(\sin x)^3 + C$

$u = \sin x$
$du = \cos x\, dx$

Substitution Rule

If $u = g(x)$ is a differentiable function with range over some interval and f is continuous on the same interval, then

$$\int f'(g(x))g'(x)\, dx = \int f(u)\, du$$

Two important points to keep in mind:

- After substitution, the integral should be in terms of one variable only
- After solving, go back to the original variable to formulate the final answer

Examples

Ex: Evaluate the integrals

(i) $\int x^2 e^{x^3}\,dx = \int e^{x^3}\cdot x^2\,dx$

$u = x^3$, $du = 3x^2\,dx$, $\frac{du}{3} = x^2\,dx$

$= \int e^u \cdot \frac{du}{3} = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C$ //

(ii) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

$u = \cos x$, $du = -\sin x\,dx$

$= \int -\frac{du}{u} = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C = \ln|\sec x| + C$ //

(iii) $\int \cos^3(x)\sin(x)\,dx$

$u = \cos x$, $du = \sin x\,dx$

$= \int u^3\,du = \frac{u^4}{4} + C = \frac{1}{4}(\cos x)^4 + C$ //

(iv) $\int e^x \sin(e^x)\,dx$

$u = e^x$, $du = e^x\,dx$

$= \int \sin u \cdot du = -\cos u + C = -\cos(e^x) + C$ //

(v) $\int x^2 \cdot \sin(x^3+1)\,dx = \int \sin(x^3+1)\cdot x^2\,dx$

$u = x^3+1$, $du = 3x^2\,dx$, $\frac{du}{3} = x^2\,dx$

$= \int \sin u \,\frac{du}{3} = \frac{1}{3}\int \sin u\,du = \frac{1}{3}(-\cos u) + C = -\frac{1}{3}\cos(x^3+1) + C$ //

Practice Examples

Ex: Evaluate the integrals

(i) $\int x\sin(x^2)\,dx = \int \sin(x^2)\,x\,dx$

$u = x^2$, $du = 2x\,dx$, $\frac{du}{2} = x\,dx$

$= \int \sin u \cdot \frac{du}{2} = \frac{1}{2}\int \sin u\,du = \frac{1}{2}(-\cos u) + C = -\frac{1}{2}\cos(x^2) + C$ //

(ii) $\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx$

$u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$, $2du = \frac{dx}{\sqrt{x}}$

$= \int \cos(\sqrt{x})\cdot\frac{dx}{\sqrt{x}} = \int \cos u \cdot 2\,du = 2\int \cos u\,du = 2\sin u + C = 2\sin(\sqrt{x}) + C$ //

(iii) $\int \frac{2x}{1+x^2}\,dx = \int \frac{du}{u}$

$u = 1+x^2$, $du = 2x\,dx$

$= \ln|u| + C = \ln|1+x^2| + C$ //

(iv) $\int \frac{1}{x\ln x}\,dx = \int \frac{1}{\ln x}\cdot\frac{dx}{x}$

$u = \ln x$, $du = \frac{1}{x}\,dx$

$= \int \frac{1}{u}\,du = \ln|u| + C = \ln|\ln|x|| + C$

Substitution Rule for Definite Integrals

If $g'(x)$ is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x)).g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Ex: Evaluate $\int_0^2 \sqrt{3x+1}\,dx$

$u = 3x+1$
$du = 3\,dx$
$\frac{du}{3} = dx$

$x=0 : u=1$
$x=2 : u=7$

$$= \int_1^7 \sqrt{u}\,\frac{du}{3}$$
$$= \frac{1}{3}\int_1^7 \sqrt{u}\,du$$
$$= \frac{1}{3}\left|\frac{u^{3/2}}{3/2}\right|_1^7$$
$$= \frac{2}{9}\left|u^{3/2}\right|_1^7 = \frac{2}{9}\left(7^{3/2}-1\right)$$ //

* Equation of semicircle with radius 'a' & center at (0,0) is $y = \sqrt{a^2-x^2}$

Examples

Ex: Evaluate the definite integrals:

(a) $\int_0^1 e^x \sin(e^x)\,dx$

$u = e^x$
$x=0 : u = e^0 = 1$
$x=1 : u = e^1 = e$
$du = e^x\,dx$

$$= \int_1^e \sin u\,du$$
$$= \left|-\cos u\right|_1^e$$
$$= (-\cos e) - (-\cos 1)$$
$$= \cos(1) - \cos(e)$$

(b) $\int_0^a x\sqrt{a^2-x^2}\,dx = \int_0^a \sqrt{a^2-x^2}\cdot x\,dx$

$u = a^2-x^2$
$du = -2x\,dx$
$\frac{du}{-2} = x\,dx$
$x=0 : u = a^2$
$x=a : u=0$

$$= \int_{a^2}^0 \sqrt{u}\cdot\frac{du}{-2}$$
$$= -\frac{1}{2}\int_{a^2}^0 \sqrt{u}\,du$$
$$= \frac{1}{2}\int_0^{a^2} \sqrt{u}\,du$$
$$= \frac{1}{2}\left|\frac{u^{3/2}}{3/2}\right|_0^{a^2}$$
$$= \frac{1}{3}\left[(a^2)^{3/2} - 0\right]$$
$$= \frac{1}{3}a^3$$ //

Examples

Ex: Use Substitution rule to evaluate the integrals:

(a) $\int_0^1 xe^{x^2}\cos(e^{x^2})\,dx$

$u = e^{x^2}$

$du = e^{x^2}\cdot 2x\,dx$

$\frac{du}{2} = x\cdot e^{x^2}dx$

$x=0: u=e^0=1$

$x=1: u=e^{1^2}=e$

$= \int_0^1 \cos(e^{x^2})\cdot \underbrace{xe^{x^2}dx}_{du}$

$= \int_1^e \cos u\,du$

$= \left|\sin u\right|_1^e$

$= \sin(e) - \sin(1)$ //

(b) $\int_e^{e^4} \frac{1}{x\log_e x}\,dx$

$u = \log_e x$

$u = \ln x$

$du = \frac{1}{x}dx$

$x=e: u=1$

$x=e^4: u=\log_e e^4 = 4\cdot 1 = 4$

$= \int_1^4 \frac{1}{u}\,du$

$= \left|\ln|u|\right|_1^4$

$= \ln|4| - \ln 1$

$= \ln|4| - 0$

$= \ln 4$ //

Practice Examples

Ex: Evaluate the following integrals:

(a) $\int \frac{\sin 2x}{1+\cos^2 x}\,dx$

$u = 1+\cos^2 x$

$= \int \frac{2\sin x\cdot\cos x}{1+\cos^2 x}\,dx$

$= \int -\frac{du}{u} = -\ln|u| + C$

$= -\ln|1+\cos^2 x| + C$ //

(b) $\int \cot x\,dx$

$u = \sin x$

$= \int \frac{\cos x}{\sin x}\,dx$

$= \int \frac{du}{u}$

$= \ln|u| + C = \ln|\sin x| + C$ //

(c) $\int e^{\cos x}\sin x\,dx$

$= \int e^u(-du)$

$= -\int e^u\,du$

$= -e^u + C$

$= -e^{\cos x} + C$ //

(d) $\int \frac{e^z+1}{e^z+z}\,dz$

$u = e^z + z$

$du = (e^z+1)\,dz$

$\int \frac{du}{u} = \ln|u| + C$

$= \ln|e^z + z| + C$ //

(e) $\int x^3\cos(x^4+2)\,dx$

$= \frac{1}{4}\sin(x^4+2) + C$ //