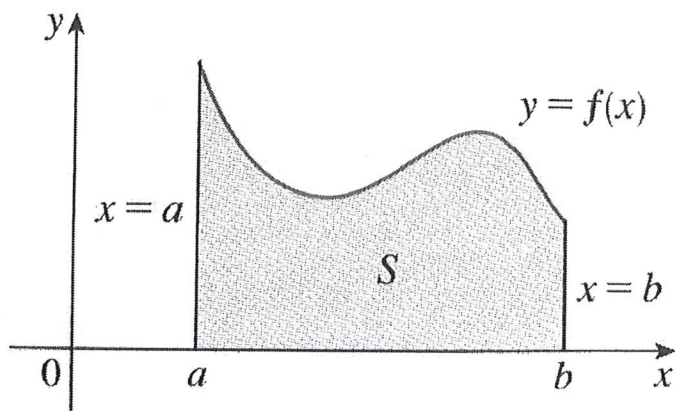


Problem

Integral calculus studies the accumulation of units as input value increases."

There are many situations in which the area under the curve will give us information about some detail related to the quantity represented by the curve.

under the curve (i.e., rate of position) the total displacement of the body.



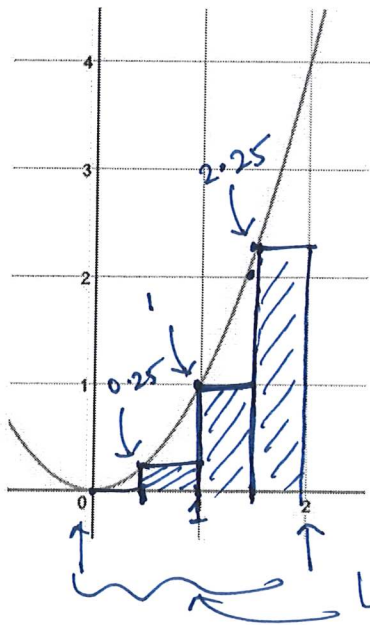
* Area under linear graphs (i.e., straight lines) can be calculated using the area of rectangle/triangle formula.

* Challenge is to calculate area under a curve boundary.

Calculating Area with Vertical Strips

Idea: Divide the area into smaller pieces and estimate the area of each piece with a simple shape (such as a rectangle).

Ex: Compute the area under $y = x^2$ on in the interval $[0, 2]$.



$$\begin{aligned} \text{Number of subrectangles} &= 4 \\ \text{Width of each subrectangle} &= \frac{2-0}{4} \\ &= \frac{2}{4} = 0.5 \end{aligned}$$

Area under the graph

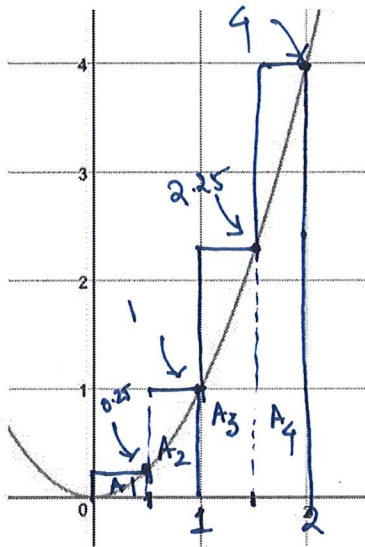
$$\begin{aligned} &\approx A_1 + A_2 + A_3 + A_4 \\ &\quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ &0 \quad 0.25 \times 0.5 \quad 1 \times 0.5 \quad 2.25 \times 0.5 \end{aligned}$$

$$= 1.75$$

Left end point approximation

Calculating Area with Vertical Strips

Right-end point approx.



Number of subrectangles = 4
 Width of each subrectangle = $\frac{2-0}{4} = 0.5$

Area under the graph

$$\approx A_1 + A_2 + A_3 + A_4$$

$$= \underbrace{0.5 \times 0.25}_{A_1} + \underbrace{0.5 \times 1}_{A_2} + \underbrace{0.5 \times 2.25}_{A_3} + \underbrace{0.5 \times 4}_{A_4}$$

$$= 3.75$$

Midpoint Approx.

$$1.75 < \text{Actual area} < 3.75$$

(lower sum) left end point approx.

right end point approx. (upper sum)

Riemann Summation

Generalization: Our goal is to determine a way to calculate areas that are enclosed by curved boundaries.

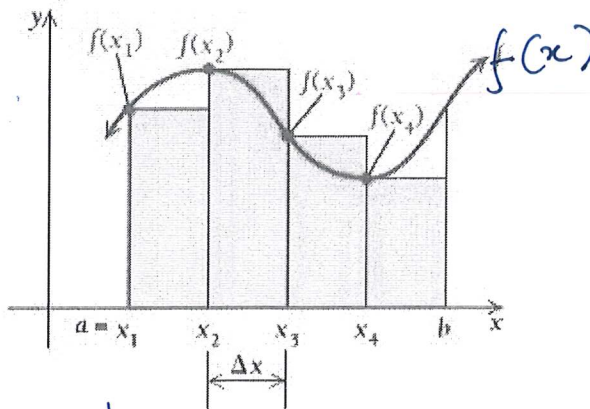
For a continuous function $f(x)$ defined over $[a, b]$, the area under the graph of $f(x)$ over the interval $a \leq x \leq b$ and above the x-axis can be approximated as the sum of the areas of the smaller rectangles.

Number of subrectangles = N ; Interval $[a, b]$

Width of each subrectangle = $\frac{b-a}{N}$

$$\Delta x = \frac{b-a}{N}$$

Area Estimation



Left Approx.

Right Approx.

Midpoint Approx.

Left/Right end-point approximation

Common details: N , $[a, b]$, $f(x)$, $\Delta x = \frac{b-a}{N}$

Left-end point Approximation
(Lower Riemann Sum)

Location of k^{th} subrectangle

$$x_k = a + (k-1)\Delta x \quad k=1, 2, \dots, N$$

Height of each subrectangle
 $= f(x_k)$

Area of k^{th} subrectangle
 $= f(x_k) \cdot \Delta x$

Area under graph $\approx \sum_{k=1}^N f(x_k) \Delta x$

Right-end point approx.

(Upper Riemann Sum)

Location of k^{th} subrectangle

$$x_k = a + k \cdot \Delta x \quad k=1, 2, \dots, N$$

Height of each subrectangle k
 $= f(x_k)$

Area of k^{th} subrectangle k
 $= f(x_k) \cdot \Delta x$

Area under graph $\approx \sum_{k=1}^N f(x_k) \Delta x$

Practice Example

Ex: Consider the graph of the function

$$f(x) = x^3 + 1$$

on the interval $[0, 2]$.

(a) Estimate the total area covered by using

4 left rectangles.

$$x_k = a + (k-1)\Delta x$$

$$= 0 + (k-1)0.5$$

$$x_k = (k-1)0.5$$

$$k=1, 2, 3, 4 \quad x_1 = 0 \rightarrow f(x_1) = 0^3 + 1 = 1$$

$$x_2 = 0.5 \rightarrow f(x_2) = 0.5^3 + 1 = 1.125$$

$$x_3 = 1 \rightarrow f(x_3) = 1^3 + 1 = 2$$

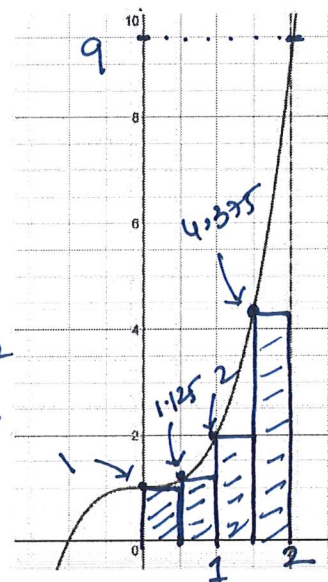
$$x_4 = 1.5 \rightarrow f(x_4) = 1.5^3 + 1 = 4.375$$

$$N=4$$

$$\Delta x = \frac{2-0}{4} = 0.5$$

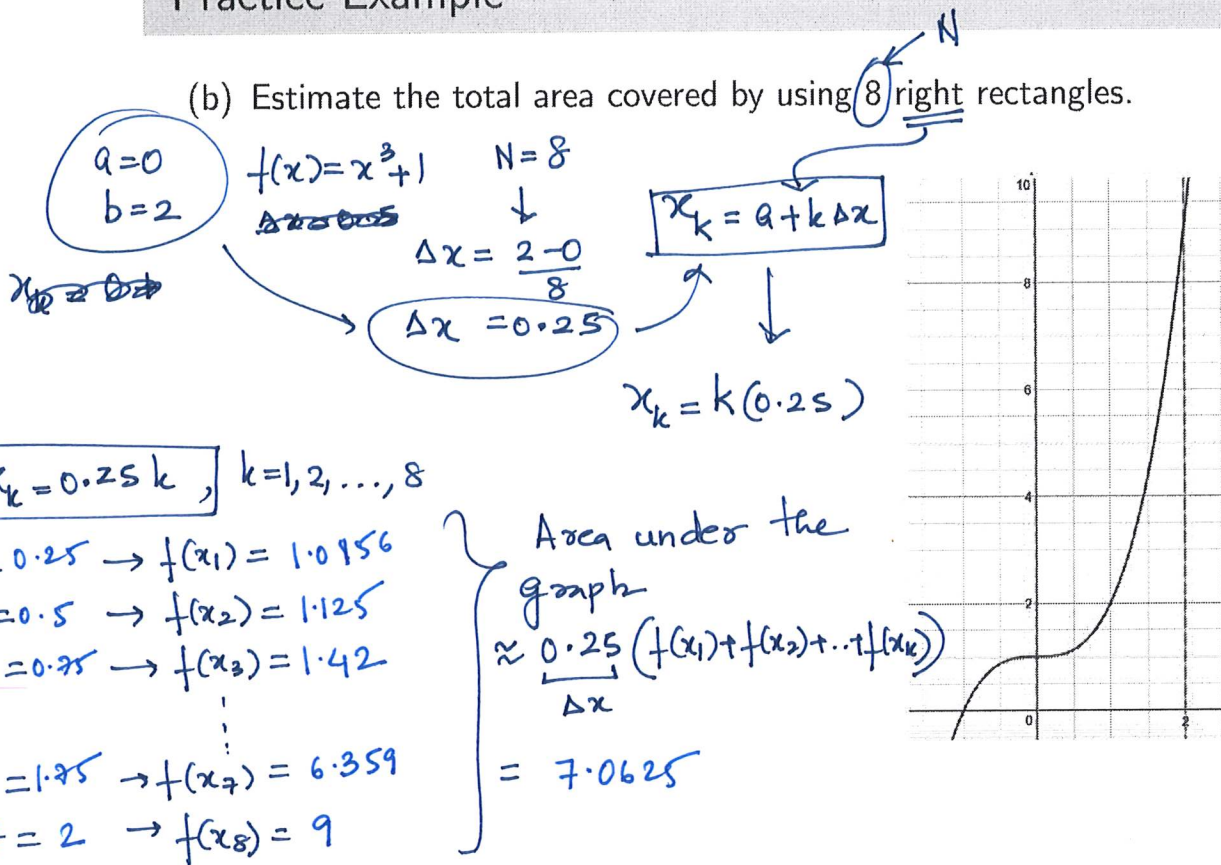
Area under graph
 $\approx 0.5 (1 + 1.125 + 2 + 4.375)$

$$= 4.3125$$



Practice Example

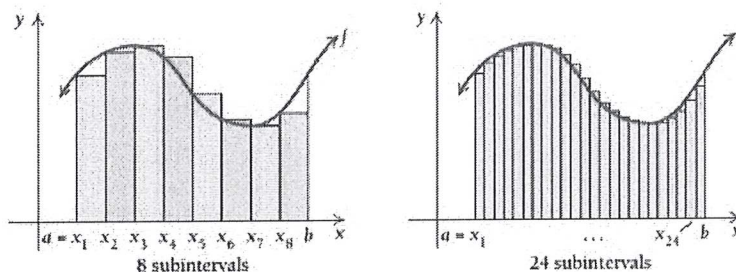
(b) Estimate the total area covered by using 8 right rectangles.



Limit of a Riemann Sum

What happens as the number of rectangles gets larger?

As N increases, the area calculation becomes more exact.



The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of the approximating rectangles.

$$A = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \cdot \Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i(\Delta x)$ or $= a + (i-1)(\Delta x)$.

↑
width of subrectangle.

↑
right ~~left~~ approx.

↑
left-end point approx.

Sigma (Summation) Notation

Σ Sigma (summation) notation is a compact way to write sums of numbers when there is a pattern in the numbers.

$$\sum_{i=m}^n a_i = a_m + a_{m+1} + \dots + a_{n-1} + a_n$$

index \rightarrow $i=m$ starting value
 n final value

Ex: Express each of the following without the summation notation.

(a) $\sum_{n=1}^4 2^n = 2^1 + 2^2 + 2^3 + 2^4$

(c) $\sum_{i=1}^3 (i^2 + i) = (1^2 + 1) + (2^2 + 2) + (3^2 + 3)$

(b) $\sum_{k=1}^5 3 = 3 + 3 + 3 + 3 + 3$

(d) $\sum_{i=1}^4 f(x_i) \Delta x = f(x_1) \Delta x + f(x_2) \Delta x + f(x_3) \Delta x + f(x_4) \Delta x$

Specialized Formula

• $\sum_{i=1}^n 1 = \underbrace{1 + 1 + \dots + 1}_{n \text{ times}} = n$

• $\sum_{i=1}^n i = 1 + 2 + \dots + n = \frac{n(n+1)}{2}$

• $\sum_{i=1}^n i^2 = 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

• $\sum_{i=1}^n i^3 = 1^3 + 2^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$

• Arithmetic of Summation Notation

▶ $\sum_{i=1}^n c \cdot a_i = c \sum_{i=1}^n a_i$

▶ $\sum_{i=1}^n (a_i \pm b_i) = \sum_{i=1}^n a_i \pm \sum_{i=1}^n b_i$

General Case: N subintervals

Ex: Suppose we subdivide the interval $[0, 2]$ under $y = x^2$ into N pieces.

~~$y = f(x) = x^2$~~ $y = f(x) = x^2$ on $[0, 2]$

Width of each subrectangle

$$\Delta x = \frac{b-a}{N} = \frac{2}{N}$$

Location of k th subrectangle (right-end point approx)

$$x_k = a + k \cdot \Delta x = 0 + k \cdot \frac{2}{N} = \frac{2k}{N}$$

Height of each subrectangle

$$= f(x_k)$$

$$= \left(\frac{2k}{N}\right)^2$$

$$= \frac{4}{N^2} k^2$$

General Case

$$\text{Area of each subrectangle} = f(x_k) \cdot \Delta x = \frac{4k^2}{N^2} \cdot \frac{2}{N} = \frac{8}{N^3} k^2$$

$$\text{Sum of the areas of } k \text{ subrectangles} = \sum_{k=1}^N f(x_k) \cdot \Delta x$$

$$S_N = \sum_{k=1}^N \frac{8}{N^3} k^2$$

$$= \frac{8}{N^3} \sum_{k=1}^N k^2 = \frac{8}{N^3} \cdot \frac{N(N+1)(2N+1)}{6}$$

$$= \frac{8}{N^2} \left(\frac{(N+1)(2N+1)}{6} \right)$$

Area under the graph

$$= \lim_{N \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} \frac{8}{N^2} (N+1)(2N+1)$$

$$= \frac{8}{6} \lim_{N \rightarrow \infty} \left(1 + \frac{1}{N}\right) \left(2 + \frac{1}{N}\right)$$

$$= \frac{8}{6} (1+0)(2+0) = \frac{8}{6} (1)(2) = \frac{8}{3}$$

exact area.

$$\frac{8}{3}$$

Practice Example

Ex: Find an expression for the area A that lies under the graph of

$$f(x) = e^{-x}$$

between $x=0$ and $x=2$ as a limit. Do not evaluate the limit.

a

b

$$\begin{aligned} \Delta x &= \frac{b-a}{N} = \frac{2}{N} \\ x_k &= a + k \cdot \Delta x = 0 + \frac{2k}{N} \\ f(x_k) &= e^{-x_k} = e^{-2k/N} \\ A_k &= f(x_k) \cdot \Delta x = \frac{2}{N} e^{-2k/N} \\ S_N &= \sum_{k=1}^N A_k = \frac{2}{N} \sum_{k=1}^N e^{-2k/N} \end{aligned}$$

Area A under the graph of $f(x)$

$$= \lim_{N \rightarrow \infty} S_N$$

$$= \lim_{N \rightarrow \infty} \frac{2}{N} \sum_{k=1}^N e^{-2k/N}$$

Practice Example

Ex: Determine a region whose area is equal to the given limit.

(a) $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \left(\frac{i}{n}\right)^3$

$$= \lim_{n \rightarrow \infty} S_n$$

$$S_n = \sum_{i=1}^n f(x_i) \Delta x$$

$$\sum \frac{1}{n} \left(\frac{i}{n}\right)^3 = \sum f(x_i) \Delta x$$

$$f(x_i) = \left(\frac{i}{n}\right)^3$$

$$\Delta x = \frac{1}{n} = \frac{b-a}{n}$$

$$f(x) = x^3$$

$$a=0 \rightarrow [0,1]$$

$$b=1$$

$$x_i = a + i \Delta x = 0 + i \cdot \frac{1}{n} = \frac{i}{n}$$

(b) $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} \sqrt{1 + \frac{3i}{n}} = \lim_{n \rightarrow \infty} S_n$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

$$f(x_i) = \sqrt{1 + \frac{3i}{n}}$$

$$\Delta x = \frac{3}{n} = \frac{b-a}{n}$$

$$a=0$$

$$b=3$$

$$x_i = a + i \Delta x = 0 + i \left(\frac{3}{n}\right) = \frac{3i}{n}$$

$$f(x_i) = \sqrt{1 + x_i} \Rightarrow f(x) = \sqrt{1+x}$$

Example

Two ways for calculating a Definite Integral (or Evaluating the Area under a graph):

- Area under Straight Lines (Linear Functions)
- Area under Curved boundaries (Non-linear Functions)

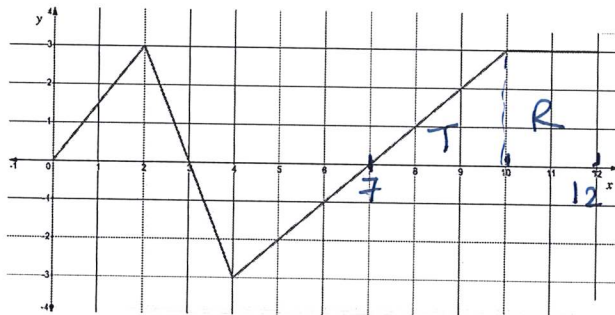
Area of triangle formula
Area of rectangle formula

Ex: Using the graph of the function $f(x)$, evaluate

(i) $\int_0^2 f(x) dx$

$$= \frac{1}{2} \times 2 \times 3 = 3$$

↑ base
↑ height



(ii) $\int_7^{12} f(x) dx$

$$= \frac{1}{2} \times 3 \times 3 + 2 \times 3 = \frac{9}{2} + 6 = 4.5 + 6 = 10.5$$

triangle
rectangle

Example

Ex: Evaluate $\int_2^4 x^2 dx$ using the Riemann Sum approach (as $N \rightarrow \infty$).

Area under graph of $f(x) = x^2$ on $[2, 4]$

$$\Delta x = \frac{b-a}{N} = \frac{4-2}{N} = \frac{2}{N}; \quad x_k = a + k \cdot \Delta x = 2 + k \cdot \frac{2}{N} = 2 + \frac{2k}{N}$$

$$f(x_k) = x_k^2 = \left(2 + \frac{2k}{N}\right)^2 = 4 + \frac{4k^2}{N^2} + \frac{8k}{N}$$

$$= \lim_{N \rightarrow \infty} \left[8 + \frac{8}{6} \left(1 + \frac{1}{N}\right) \left(2 + \frac{1}{N}\right) \right] = 8 + \frac{8}{6} \cdot 2 + 8 + 8 \left(1 + \frac{1}{N}\right) = 18.67$$

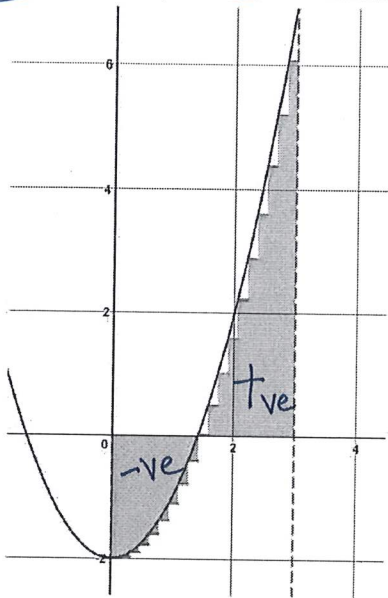
$$A_k = f(x_k) \cdot \Delta x = \frac{2}{N} \left(4 + \frac{4k^2}{N^2} + \frac{8k}{N}\right) = \frac{8}{N} + \frac{8k^2}{N^3} + \frac{16k}{N^2}$$

$$\int_2^4 x^2 dx = \text{Exact area} = \lim_{N \rightarrow \infty} \sum_{k=1}^N A_k = \lim_{N \rightarrow \infty} \sum_{k=1}^N \left(\frac{8}{N} + \frac{8k^2}{N^3} + \frac{16k}{N^2} \right) = \lim_{N \rightarrow \infty} \frac{8}{N} \sum 1 + \frac{8}{N^3} \sum k^2 + \frac{16}{N^2} \sum k$$

$$= \lim_{N \rightarrow \infty} \left(\frac{8}{N} \cdot N + \frac{8}{N^3} \cdot \frac{N(N+1)(2N+1)}{6} + \frac{16}{N^2} \cdot \frac{N(N+1)}{2} \right)$$

Signed Area

Definite integral corresponds to signed area, that is the area is positive above x-axis and negative below x-axis.



For application based problems, we would artificially introduce (-)ve sign to make the area calculation positive sometimes!

Example,

Velocity \rightarrow Displacement
 \rightarrow Distance.

Examples

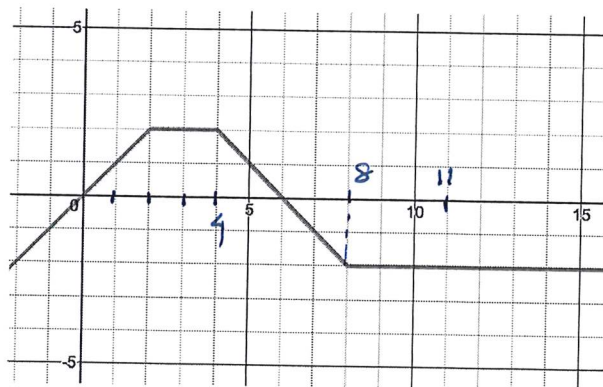
Ex: Using the graph of the function $f(x)$, evaluate

(i) $\int_0^1 f(x) dx = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

(ii) $\int_0^4 f(x) dx = \frac{1}{2} \times 2 \times 2 + 2 \times 2 = 6$

(iii) $\int_0^8 f(x) dx = 6$

(iv) $\int_0^{10} f(x) dx = 2$



(v) For what value(s) of x is $\int_0^x f(x) dx = 0$? $x = 0, 11$

(vi) For what value(s) of x is $\int_0^x f(x) dx$ negative? $x > 11$

Practice Example

Ex: Evaluate $\int_0^4 (x^3 + 1) dx$ using the Riemann Sum approach (as $N \rightarrow \infty$).

$$\frac{4-0}{N} = \frac{4}{N}$$

$$a + k \cdot \Delta x$$

$$= 0 + k \cdot \frac{4}{N}$$

$$= \frac{4k}{N}$$

$$= x_k^3 + 1$$

$$= \frac{64}{N^3} k^3 + 1$$

$$f(x_k) \cdot \Delta x$$

$$= \left(\frac{64}{N^3} k^3 + 1 \right) \frac{4}{N}$$

$$= \frac{256}{N^4} k^3 + \frac{4}{N}$$

$$\int_0^4 (x^3 + 1) dx = \lim_{N \rightarrow \infty} \sum_{k=1}^N f(x_k) \Delta x$$

$$= \lim_{N \rightarrow \infty} \sum_{k=1}^N \left(\frac{256}{N^4} k^3 + \frac{4}{N} \right)$$

$$= \lim_{N \rightarrow \infty} \left[\frac{256}{N^4} \sum_{k=1}^N k^3 + \frac{4}{N} \sum_{k=1}^N 1 \right]$$

$$= \lim_{N \rightarrow \infty} \left[\frac{256}{N^4} \cdot \frac{N^2(N+1)^2}{4} + \frac{4}{N} \cdot N \right]$$

$$= \lim_{N \rightarrow \infty} \left[\frac{256}{4} \left(1 + \frac{1}{N} \right)^2 + 4 \right]$$

$$= 64 (1+0)^2 + 4$$

$$= 68 //$$

Properties of Definite Integrals

Arithmetic of Integration: Let a, b and A, B, C be real numbers. Let the functions $f(x)$ and $g(x)$ be integrable on an interval that contains a and b . Then

- $\int_a^b C dx = C(b-a)$, where C is any constant.

- $\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$

- $\int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$

- $\int_a^b cf(x) dx = c \cdot \int_a^b f(x) dx$, where c is any constant.

Examples

Ex: Use the properties of Definite integrals to evaluate:

Given: $\int_0^1 x^2 dx = \frac{1}{3}$

(i) $\int_0^1 (4 + 3x^2) dx = \int_0^1 4 dx + 3 \int_0^1 x^2 dx$

$= 4 \int_0^1 1 \cdot dx + 3 \int_0^1 x^2 dx$ ← Given.

$= 4 \cdot (1 \times 1) + 3 \cdot \frac{1}{3}$

$= 4 + 1 = 5$

(ii) If $\int_2^3 f(x) dx = -1$ and $\int_2^3 g(x) dx = 5$, $\int_2^3 (6f(x) - 3g(x)) dx =$

$\int_2^3 6f(x) - 3g(x) dx = 6 \int_2^3 f(x) dx - 3 \int_2^3 g(x) dx$

$= 6 \times (-1) - 3 \times 5$

$= -6 - 15 = -21 //$

Properties of Definite Integral

Arithmetic for the Domain of Integration: Let a, b, c be real numbers. Let the function $f(x)$ be integrable on an interval that contains a, b and c .

Then

• $\int_a^a f(x) dx = 0$ → Collapse the domain of int.

• $\int_a^b f(x) dx = -\int_b^a f(x) dx$ ← reverse the domain of integration

• $\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx$ ← Break the domain into two or more sections.

For symmetric functions,

• for an even function, $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$

Symm. about y-axis → $f(-x) = f(x)$

• for an odd function, $\int_{-a}^a f(x) dx = 0$

Symm. about origin → $f(-x) = -f(x)$

Examples

Ex: If $\int_2^6 f(x) dx = 7$ and $\int_2^3 f(x) dx = 5$, then

$$\int_3^6 f(x) dx = ?$$

$$\int_2^6 f(x) dx = \int_2^3 f(x) dx + \int_3^6 f(x) dx = 5 + \int_3^6 f(x) dx$$

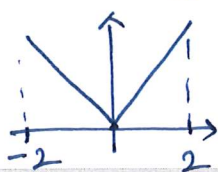
$$7 = 5 + \int_3^6 f(x) dx \Rightarrow \int_3^6 f(x) dx = 2$$

Ex: Assuming that for any real number $b > 0$, $\int_0^b x dx = \frac{b^2}{2}$,

evaluate $\int_{-2}^3 |x| dx$.

Given!

$$\int_0^b x dx = \frac{b^2}{2}$$



$$\int_{-2}^3 |x| dx = \int_{-2}^2 |x| dx + \int_2^3 |x| dx$$

even function

$$= \int_{-2}^0 |x| dx + \int_0^3 |x| dx$$

$$= \int_{-2}^0 -x dx + \int_0^3 x dx$$

$$= \int_0^2 x dx + \int_0^3 x dx$$

$$= 2 \int_0^2 |x| dx + \int_2^3 x dx$$

$$= 2 \int_0^2 x dx + \int_2^3 x dx$$

$$= 2 \cdot \frac{2^2}{2} + \frac{3^2}{2} - \frac{2^2}{2}$$

$$= 4 + \frac{9}{2} = 4 + 4.5 = 8.5$$

Comparison Principles

For $a \leq x \leq b$,

• If $f(x) \geq 0$, then $\int_a^b f(x) dx \geq 0$.

• If $f(x) \geq g(x)$, then $\int_a^b f(x) dx \geq \int_a^b g(x) dx$.

• If $m \leq f(x) \leq M$, then $m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$.

Example

Ex: Give the upper and lower bounds on the value of $\int_0^1 e^{-x^2} dx$.

$$\int_0^1 e^{-x^2} dx$$

$$0 \leq x \leq 1$$

$$0 \leq x^2 \leq 1$$

$$0 \geq -x^2 \geq -1$$

$$-1 \leq -x^2 \leq 0$$

$$e^{-1} \leq e^{-x^2} \leq e^0$$

$$0.367 = \frac{1}{e} \leq e^{-x^2} \leq 1$$

$$a=0$$

$$b=1$$

$$b-a=1-0=1$$

Comparison Principle: If $m \leq f(x) \leq M$, then $m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$

$$\frac{1}{e} \leq \int_0^1 e^{-x^2} dx \leq 1$$

Examples

Ex: Use the Comparison principles to estimate the value of the integral

$$\int_{0=a}^{\pi/3=b} \sqrt{\cos x} dx$$

$$\left. \begin{array}{l} a=0 \\ b=\pi/3 \end{array} \right\} b-a=\pi/3$$

$$f(x) = \sqrt{\cos x}$$

$$g(x) = \cos x$$

$$g(0) = \cos 0 = 1$$

$$g(\pi/3) = \cos \pi/3 = \frac{1}{2}$$

$$\frac{1}{\sqrt{2}} \leq \sqrt{\cos x} \leq 1, \quad 0 \leq x \leq \pi/3$$

Using comparison principle,

$$\frac{\pi}{3\sqrt{2}} \leq \int_0^{\pi/3} \sqrt{\cos x} dx \leq \frac{\pi}{3}$$

$$0.7404 \leq \int_0^{\pi/3} \sqrt{\cos x} dx \leq 1.04719$$